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Chaos Theory and the Lorenz Equation: History, Analysis, and Application

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1.0 Introduction

The purpose of this paper is to provide the reader with an introduction to the concepts of chaos theory and sensitive dependence on initial conditions. One of the first and most famous equations regarding these two topics, the Lorenz Equation, will be discussed. The equation's history, properties, and graphical interpretations will be examined. This paper concludes with a brief discussion of chaos in our world today.

To begin the study of chaos and sensitive dependence on initial conditions, one must define these terms. "Chaos can be described as long term, aperiodic behaviour that exhibits sensitive dependence on initial conditions. Sensitive dependence on initial conditions implies that nearby trajectories diverge exponentially fast over time," (Strogatz 320). This means if one begins with two near identical states, they will arrive at two drastically different states after iterating them over some chaotic function. Mathematicians of the 1800s James Maxwell and Henri Poincare developed ideas regarding sensitive dependence on initial conditions, but it was not until Edward Lorenz that formal mathematics was used to explore this new and exciting field.

2.0 Edward Lorenz

The Lorenz Equation was named for its inventor Edward Norton Lorenz, an American mathematician and meteorologist. Born in 1918, he studied mathematics at Dartmouth and Harvard, before breaking from his work to serve as a weather forecaster for the Air Force during World War II. As the war concluded, he entered the Massachusetts Institute of Technology (MIT), where he earned a doctoral degree in meteorology. He was immediately hired as staff at MIT and was made an associate professor in 1955 (Telegraph).

In the mid-1900s, the field of meteorology was still very much in its infancy. Lorenz programmed an existing computer, the Royal McBee, to aide in his research of atmosphere equations and forecasting. His program allowed him control the initial conditions of a weather system based on 12 differential equations. Furthermore, his program could perform up to sixty calculations per second, and was able to run for a very long time without stopping. Lorenz would input data and his program would return the resulting weather and atmosphere information (Elert, Telegraph).

3.0 The Lorenz Equation

The idea behind the Lorenz Equation came in 1961, when Lorenz ran his program with data rounded off from a previous experiment. He recalls the moment he realised the 'chaos' present in weather systems:

> "I typed in some of the intermediate conditions which the computer had printed out as new initial conditions to start another computation and then went out for a while. Afterwards, I found that the solution was not the same as the one I had before. I soon found that the reason was that the numbers I had typed in were not the same, but were rounded off numbers. The small difference between something retained to six decimal places and rounded off to three had amplified in the course of two months of simulated weather until the difference was as big as the signal itself. And to me this implied that if the real atmosphere behaved in this method then we simply couldn't make forecasts two months ahead. The small errors in observation would amplify until they became large." (Peitgen)

This occurrence would become the basis for the Lorenz Equation. The fact that two weather conditions, which differ by less than 0.1%, can produce drastically different results underscores how chaos and sensitive dependence on initial conditions are integrated into meteorology.

In March 1963, Lorenz wrote that he wanted to introduce, "ordinary differential equations whose solutions afford the simplest example of deterministic non periodic flow and finite amplitude convection," (Lorenz 134). In his paper, he examines the work of meteorologist Barry Saltzman and physicist John Rayleigh while incorporating several physical phenomena (Bradley, Viswanath). Lorenz found that when applying the Fourier Series to one of Rayleigh's convection equations that, "...all except three variables tended to zero, and that these three variables underwent irregular, apparently non periodic functions," (Lorenz 135). He then used these variables to construct a simple model based on the 2-dimensional representation of the earth's atmosphere. The following is known as the Lorenz Equation:

$$\frac{dx}{dt} = \dot{X} = \sigma(y - x)$$

$$\frac{dy}{dt} = \dot{Y} = \rho x - y - xz$$

$$\frac{dz}{dt} = \dot{Z} = xy - \beta z$$

3.1 Parameter Explanation

Here x, y, z do not refer to coordinates in space (Gulick 276). In fact, x represents the convective overturning on the plane, while y and z are the horizontal and vertical temperature variation respectively. The parameters of this model are σ, which represents the Prandtl number, or the ratio between the fluid viscosity to its thermal conductivity, ρ, which represents the difference in temperature between the top and bottom of the atmosphere plane, and β, which is the ratio of the width to the height of the plane (Bradley, Gulick). Lorenz found the values of $\sigma = 10$ and $\beta = 8/3$, and initial conditions of $(x_0, y_0, z_0) = (0, 1, 0)$ to be the best representation of the earth's atmosphere (Lorenz 136-137). For this project, we assume these values.

3.2 Fixed Points

The following proof is based on Record 2003 and Rothmayer 1993

While Lorenz found chaos to be a large factor in meteorology, the equation he created does not exhibit chaos for all parameters. In fact, there are many parameter values where the function is stable and contain fixed points. We will now explore how Lorenz came to realize fixed points of his system, as well as for what values the equation exhibits chaos.

To find fixed points of the Lorenz Equation, we will first solve for its equilibria. To find these equilibrium points, we will set $\dot{X}$, $\dot{Y}$, and $\dot{Z}$ to 0.

$$\sigma(\text{y - x}) = 0 \ (1)$$

$$\rho\text{x - y - xz} = 0 \ (2)$$

$$\text{xy - }\beta\text{z} = 0 \ (3)$$

Solving:

$$\text{x} = \text{y} \ (1)$$

$$\text{y} = \rho\text{x - xz} \ (2)$$

$$\text{x} = \text{x}(\rho\text{ - z}) \Rightarrow \text{z} = \rho\text{ - 1} \ (1)\ (2)$$

$$x^2 \text{ - } (\rho\text{ - 1})\beta = 0 \ (1)(3)$$

$$x = y = \pm \sqrt{\beta(\rho - 1)}$$

Our three equilibrium points are

$$(0, 0, 0), K_1 = (- \sqrt{\beta(\rho-1)}, - \sqrt{\beta(\rho-1)}, \rho - 1), K_2 = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1)$$

The behaviour of the Lorenz Equation is complex, so we consider cases of the parameters. We established that Lorenz preferred $\sigma = 10$ and $\beta = 8/3$, and thus, we will only concern ourselves with ρ.

Case 1: 0<ρ<1

(0, 0, 0) yields the only real fixed point of the equilibrium points. There is no chaos when $0 < \rho < 1$

Case 2: $\rho = 1$

We identify $\rho = 1$ as a bifurcation point, as the other two equilibrium points will appear when $\rho > 1$. There is no chaos when $\rho = 1$

Case 3: $\rho > 1$

We have two new fixed points $K_1 = (- \sqrt{\beta(\rho-1)}, - \sqrt{\beta(\rho-1)}, \rho - 1)$, $K_2 = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1)$

We will need to check the stability of these points by linearizing the Lorenz Equation and finding its eigenvalues. A point is stable when its eigenvalues are all negative. We will linearize the system near an already established equilibrium point from above, call it $(\overline{X}, \overline{Y}, \overline{Z})$ using the Jacobian Matrix. This gives

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - \overline{Z} & -1 & \overline{X} \\ \overline{Y} & \overline{X} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

To get the eigenvalues of the above 3 x 3 matrix (call it A), we solve det(A - λ I) = 0

This yields

$$det \begin{bmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - \overline{Z} & -1 - \lambda & \overline{X} \\ \overline{Y} & \overline{X} & -\beta - \lambda \end{bmatrix} = 0$$

$$\lambda^3 + (\beta + \sigma + 1)\lambda^2 \ + (\beta + \beta\sigma + \sigma - \rho\sigma + \sigma\overline{Z} + \overline{X}^2)\lambda + \beta\sigma(1-\rho) + \sigma(\overline{XY} + \overline{X}^2 + \beta\overline{Z}) = 0(4)$$

If we take $(\overline{X}, \overline{Y}, \overline{Z})$ to be the equilibrium point (0, 0, 0), we get

$$\lambda^3 + (\beta + \sigma + 1)\lambda^2 \ + (\beta + \beta\sigma + \sigma - \rho\sigma)\lambda + \beta\sigma(1-\rho) = 0$$

-β is a solution, so we can factor to get

$$(\lambda + \beta)(\lambda^2 + (\sigma + 1)\lambda + \sigma(1-\rho)) = 0$$

Thus the eigenvalues are

$$\lambda_1, \lambda_2 = \frac{-\sigma - 1 \pm \sqrt{(\sigma+1)^2 + 4\sigma(\rho - 1)}}{2}, \lambda_3 = -\beta$$

When $\rho > 1$ however, $\lambda_1 > 0$ λ_2, $\lambda_3 < 0$ and (0, 0 , 0) is not stable and thus not a fixed point for $\rho > 1$

If we take $(\overline{X}, \overline{Y}, \overline{Z})$ to be either K_1 or K_2 and plug them into (4), we end up with eigenvalues

$$\mu^3 + (\beta + \sigma + 1)\mu^2 \ + (\sigma + \rho)\beta\mu + (1-\rho)2\sigma\beta = 0$$

All three eigenvalues μ_1, μ_2, μ_3 will be negative when

$$\rho < \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1} = \rho_c$$

Substituting $\sigma = 10$, $\beta = 8/3$, we get $\rho < \frac{470}{19} \approx 24.74 = \rho_c$

Thus, K_1, K_2 are stable and fixed points when $1 < \rho < 24.74$. When $\rho \geq 24.74$, not all of μ_1, μ_2, μ_3 will be negative, and K_1, K_2 will not be stable and thus not fixed points.

At $\rho > \rho_c$, the three equilibrium points ((0, 0, 0), K_1, K_2) are unstable and are not fixed points. They do not approach infinity, but rather enter a region around the origin. This is where we begin to see chaotic behaviour of the Lorenz Equation.

To recap the fixed points of the Lorenz Equation are

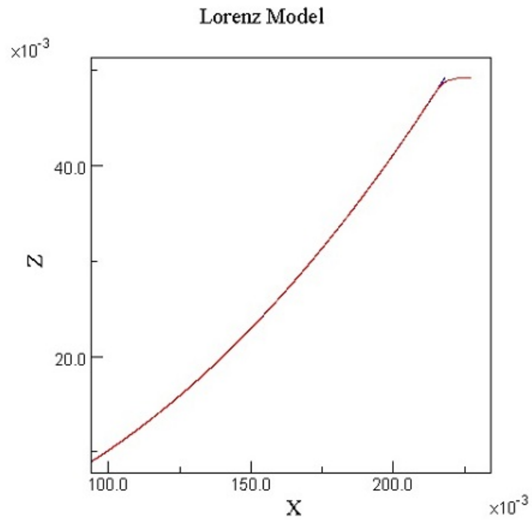
ρ	Fixed Points
[0-1]	(0, 0, 0)
(1, 24.74)	K_1, K_2
[24.74-30.1)	None, chaos occurs
[30.1 - ∞)	intermittency (not proven)

3.3 The Lorenz Attractor

As shown above, when $24.74 \leq \rho < 30.1$ the Lorenz Equation displays chaos. This condition on ρ gives the equation a 'nickname': The Lorenz Attractor. The Lorenz Attractor is a strange attractor, which means the equation is non-periodic, as thus never repeats itself. Strange attractors are also coupled with the notion of chaos and sensitive independance on initial conditions, in that one cannot predict where on the attractor the system will be in the future.

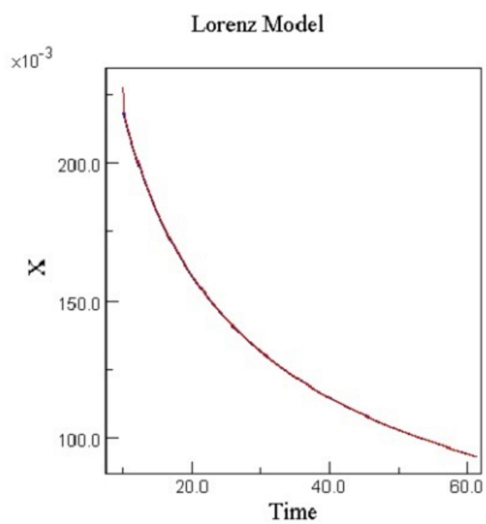
4.0 Graphical Interpretations of the Lorenz Equation

The following graphs are based on the Lorenz Equation using initial conditions $(x_0, y_0, z_0) = (0, 1, 0)$, $\sigma = 10$, $\beta = 8/3$, and time = 60 seconds; only ρ is changed. Two slightly different starting points (one blue, the other red) and their subsequent orbits are plotted on the *x-z* plane as well as the *x-time* plane. It should be noted that another property of the Lorenz Equation, is that it is deterministic, which means each set of initial conditions produces a unique graph (Carriuolo 2006). One would expect different graphs if we had used the initial conditions (x_0, y_0, z_0)=(1, 0, 1), though our results and conclusions would remain unchanged



xz.jpg

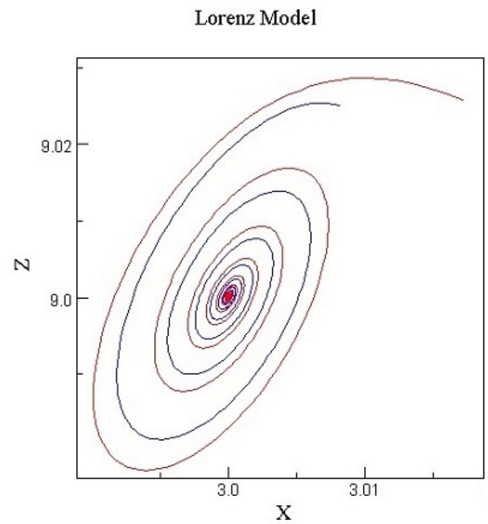
Figure 1: XZ plane, $\rho = 1$



time.jpg

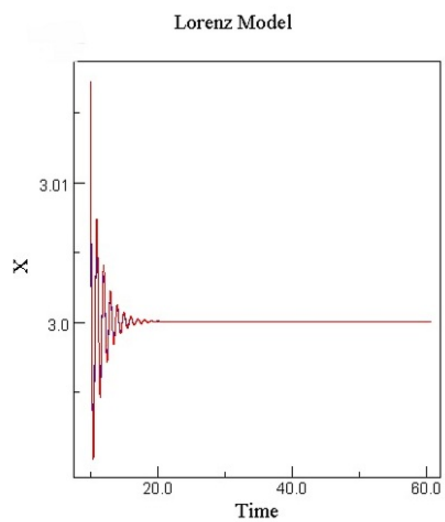
Figure 2: Time vs X, $\rho = 1$

The red and blue orbits appear to behave similarly over time; their paths are very much the same. Accordingly, no chaos is present in this equation. This is to be expected as $\rho = 1$ produces a stable system.



xz.jpg

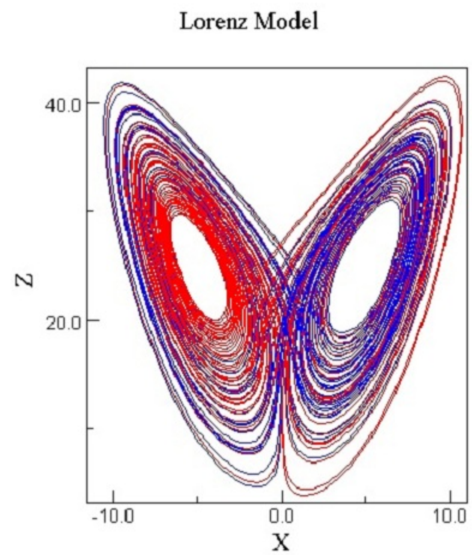
Figure 3: XZ plane, $\rho = 10$



time.jpg

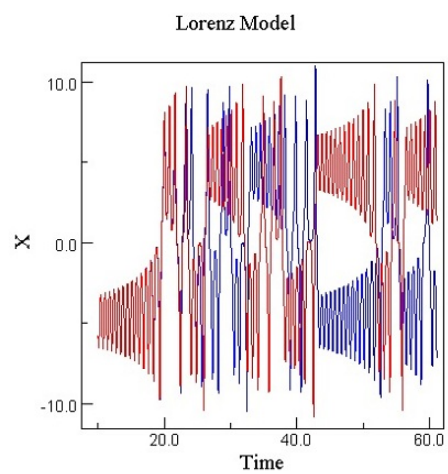
Figure 4: Time vs X, $\rho = 10$

Again, the blue and red orbits behave very similarly over time and over much iteration; no chaos is present. This is to be expected as $\rho = 10 < 24.74$



xz.jpg

Figure 5: XZ plane, $\rho = 26$



time.jpg

Figure 6: Time vs X, $\rho = 26$

These two graphs of the Lorenz Equation demonstrate chaos. Small, initial distances between the blue and red points lead to large, unpredictable distances between them after numerous iterations on the x-z plane and over time . Hence, sensitive dependence on initial conditions and chaos are present in this equation. This is expected, as $24.74 \leq \rho < 30.31$.

It is difficult to notice on the two-dimensional x-z plane, but orbits on Figure 5 (and for all Lorenz Attractor graphs) do not intersect. The red orbit never overlaps itself nor the blue orbit and vice versa. This emphasizes the idea that these orbits are non-periodic. If the orbits were to cross one another, cycles would exist, and therefore chaos would be absent (Gulick 282).

Lorenz himself was able to produce the x-z plane and time series images on his computer. In conclusion to his famous paper, he declared, "when our result concerning the instability of non periodic flow are applied to the atmosphere, they indicate that prediction of the sufficiently distant future is impossible by any method," (Lorenz 141). Here, Lorenz states that sensitive dependence on initial conditions make it difficult to predict the weather, or any chaotic system for that matter.

5.0 The Butterfly Effect

One can see that Figure 5 resembles a butterfly, and thus, the Lorenz Attractor and chaos in general are nicknamed, "The Butterfly Effect." This draws upon Lorenz's findings that two seemingly identical weather systems could produce two very different weather systems in the near future. Thus, a butterfly flapping its wings could alter the atmosphere ever so slightly, so as to deviate from the initial conditions, and accordingly alter the course of weather forever. Lorenz first used the example of a seagull's wings, though the analogy has morphed into using a butterfly (Telegraph).

6.0 Applications of Chaos

Chaotic systems have many applications, in particular to nonlinear equations. Chaotic systems can be used to explain topics in engineering, geography, and even the stock market. In 1991, Edgar E Peters, wrote, "for over 30 years, there were many people who thought the stock market follows regular cycles. Recent research, however, suggests the S&P 500 has non periodic cycles, governed by attractors." Peters goes on to formulate several equations proving his claim and explains how one can determine whether a system is a chaotic. To do so, one must use system equations to construct a phase space (this can be very difficult). Also, this phase space needs to have fractal dimension and sensitive dependence on initial conditions. Peters states that the need for chaotic systems to exist is because linear financial models have failed many times. As with Lorenz's conclusion about weather prediction, Peters states, "long range economic forecast is not feasible beyond a short time frame," (Peters 62). This example highlights one of the many ways our lives and the world around us are filled with chaos.

7.0 Conclusion

Though discovered by accident, the Lorenz Equation has had a significant contribution to mathematics and many other disciplines. In particular, the Lorenz Equation helped pioneer the study of chaos and sensitive dependence on initial conditions. This equation behaves like any other family of equations, in that it has fixed and bifurcation points, which can be graphed accordingly. Unlike other families, however, this equation is chaotic for certain parameter values. When it is chaotic, it is known as the Lorenz Attractor, which has certain chaotic properties. The Lorenz Equation has made quantifying chaos possible which has inspired many mathematicians to research and study chaos.

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Appendix A – Lorenz Equation Graphs

The figures in this project were made using the website `<http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson1/Demo8.html>`