# Correspondence Between Fractal-Wavelet Transforms and Iterated Function Systems With Grey Level Maps


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**Abstract**

It is well known that the action of a "Fractal Transform" or (Local) Iterated Function System with Grey Level Maps (IFSM) on a function $f(x)$ induces a very simple mapping on its expansion coefficients $c_{ij}$ in the Haar wavelet basis. This is the basis of the "discrete fractal-wavelet transform": subtrees of the wavelet coefficient tree are scaled and copied to lower subtrees. Such transforms, which we shall also refer to as IFSW - IFS on wavelet coefficients - have been introduced into image processing with other (compactly supported) wavelet basis sets in an attempt to remove the blocking artifacts in the standard IFS block encoding algorithms. Although not as straightforward as in the Haar case, we show that there is a relationship between such wavelet transforms and IFSM. In fact, for most such transforms, there is an equivalent IFSM, which provides a further mathematical basis for their use in image processing. We also present results for the case of periodized wavelets, the usual implementation in image processing. Finally, we prove some results on the fractal dimension of the graph of an attractor of IFSM or IFSW operators.

# 1 Introduction

Recently, a number of workers [D, FV2, KMK, S, vW] (to name a few) have independently devised "discrete fractal-wavelet transforms" (DFWT) which involve an IFS-type transformation on the wavelet expansion coefficients of signals and images. In the case of Haar wavelet expansions, as was shown in [FV2], there is a very simple connection between these wavelet transforms and (Local) Iterated Function Systems with Grey Level Maps (IFSM). Due to the nonoverlapping nature of the Haar wavelet functions, the action of the IFSM automatically translates into a simple scaling and copying of wavelet coefficient subtrees onto lower subtrees. With suitable restrictions on the scaling coefficients $\alpha_i$, the transformation is contractive in $l^2(\mathbb{N})$. The wavelet expansion coefficients $c_{ij}$ of a signal or image are then stored to a prescribed level $k^*$ of refinement, along with a set of codes describing the "parent-child" subtree pairs and their corresponding "optimal" parameters $\alpha_i$. Iteration of the fractal-wavelet transform then generates higher resolution coefficients $c_{ij}$, $i > k^*$. To repeat: In the case of Haar wavelets, this procedure in wavelet coefficient space is equivalent to the action of an IFSM in "pixel space". It also illustrates the fact that fractals and wavelet bases share the fundamental property of scaling.

Such discrete wavelet transforms were then applied to expansion coefficients for generalized (non-Haar) systems of compactly supported wavelets. In these cases, the supports of contiguous wavelets overlap. A major motivation for the generalization of this method was the attempt to reduce the blockiness exhibited by the usual fractal ( = Haar) block-encoding (see [D, KMK, vW]).

The purpose of this paper is to show that there is a connection between wavelet transforms for generalized wavelets and IFS-type methods on function spaces. However, the connection is not quite as straightforward as in the Haar case. In Section 2, we consider the case of nonperiodized wavelets and show that the discrete wavelet transform is equivalent to a recurrent IFSM. In Section 3, we consider *periodized* wavelets which are almost exclusively used in image processing. The situation is slightly more complicated but a correspondence to IFSM is shown. In Section 4, we briefly present some results on the fractal dimension of attractors for IFSM and IFSW.

We now conclude this section with a brief review of the basic ideas of IFSM. An $N$-map Iterated Function System with Grey Level Maps (IFSM) is defined by [FV1]:

1. **The IFS component:** $\mathbf{w} = \{w_1, w_2, \ldots, w_N\}$ where each $w_i : X \to X$ is a contraction and

2. **The grey level component:** $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$ where each $\phi_i : \mathbb{R} \to \mathbb{R}$ is Lipschitz.

Here $(X, d)$ is a complete metric space, the "pixel space", typically $[0, 1]$ or $[0, 1]^2$

with Euclidean metric. Associated with an IFSM $(\mathbf{w}, \Phi)$ is an operator $T$, the *fractal transform*, whose action on a suitable space of functions $\mathcal{F}(X)$ is given by

$$T(f)(x) = \sum_i \phi_i \left( f(w_i^{-1}(x)) \right), \qquad f \in \mathcal{F}(X). \tag{1}$$

For simplicity, the both the $w_i$ and the $\phi_i$ are often assumed to be affine so that $w_i(x) = s_i x + a_i$ and $\phi_i(t) = \alpha_i t + \beta_i$.

Another interesting and special case of an IFSM operator is the following:

$$T(f)(x) = \theta(x) + \sum_i \alpha_i f(w_i^{-1}(x)). \tag{2}$$

Here, the function $\theta(x)$ acts as a "condensation" function for the IFSM.

Under suitable conditions on the $w_i$ and the $\phi_i$, the operator $T$ is contractive in $\mathcal{F}(X)$ (for example, $L^p(X)$) and so has a unique fixed point $\bar{u} = T\bar{u}$ (see [FV1]).

Let $\{q_i(x)\}$ be an orthonormal basis of $L^2(X)$. Given an IFSM operator $T$ there exists a corresponding operator $M$ on the sequence space $l^2(I\!N)$ (viewed as sequences of coefficients in expansions of functions with respect to the $q_i$). The following figure illustrates this procedure:

$$\begin{tikzcd}
L^2(X) \arrow[r, "\cong"] \arrow[d, "T"'] & l^2(I\!N) \arrow[d, "M"] \\
L^2(X) \arrow[r, "\cong"] & l^2(I\!N)
\end{tikzcd}$$

In the event that the IFSM $T$ is affine or linear, then the operator $M$ will also be affine or linear. If $T$ is contractive in $\mathcal{L}^2(X)$, then $M$ is contractive in $l^2(I\!N)$. (See [FV2] for further discussion.) As mentioned above, when the orthonormal basis is the Haar wavelet basis $\psi_{ij}$, then the operator $M$ becomes a mapping of wavelet coefficient subtrees to lower subtrees. Such mappings will be introduced in the next section.

# 2 Fractal-Wavelet Transforms (IFSW) and Their Correspondence With IFSM

For simplicity, we restrict our attention to the one-dimensional case $X = [0, 1]$. The results extend in a straightforward manner to the important case $X = [0, 1]^2$

(images). Let $\phi$ be a scaling function with which is associated a multiresolution analysis of $\mathcal{L}^2(\mathbf{R})$, and let $\psi$ be the associated "mother wavelet" function from which are derived the functions

$$\psi_{mn}(x) = 2^{m/2}\psi(2^m x - n), \quad m, n \in \mathbf{Z}, \tag{3}$$

which form an orthonormal basis of $\mathcal{L}^2(\mathbf{R})$. In particular we shall be concerned with functions $f \in \mathcal{L}^2(\mathbf{R})$ which admit wavelet expansions of the form (in the $\mathcal{L}^2$ sense)

$$f(x) = b_{00}\phi(x) \ + \sum_{i \geq 0, 0 \leq j \leq 2^i - 1} c_{ij}\psi_{ij}(x), \tag{4}$$

where $b_{00} =< f, 1 >$ and $c_{ij} =< f, \psi_{ij} >$. We are interested in wavelets $\psi$ having compact support on $\mathbf{R}$ (implying, in turn, that $f$ has compact support). The expansion coefficients are conveniently displayed in the form of an *infinite* wavelet tree:

<table>
<tr><td colspan="8" align="center">$b_{00}$</td></tr>
<tr><td colspan="8" align="center">$c_{00}$</td></tr>
<tr><td colspan="4" align="center">$c_{10}$</td><td colspan="4" align="center">$c_{11}$</td></tr>
<tr><td colspan="2" align="center">$c_{20}$</td><td colspan="2" align="center">$c_{21}$</td><td colspan="2" align="center">$c_{22}$</td><td colspan="2" align="center">$c_{23}$</td></tr>
<tr><td>$B_{30}$</td><td>$B_{31}$</td><td>$B_{32}$</td><td>$B_{33}$</td><td>$B_{34}$</td><td>$B_{35}$</td><td>$B_{36}$</td><td>$B_{37}$</td></tr>
</table>

Note that each entry $B_{ij}$ represents a tree of infinite length: we shall also refer to such a tree with apex $c_{ij}$ as the *block* $B_{ij}$. We now consider some IFS-type operations on this wavelet tree.

**Example 1:** Using the above notation, consider the following transformation:

$$M \ : \ B_{00} \ \to \ \begin{array}{|c|c|} \hline \multicolumn{2}{|c|}{c_{00}} \\ \hline \alpha_0 B_{00} & \alpha_1 B_{00} \\ \hline \end{array} \quad , \quad |\alpha_i| < \frac{1}{\sqrt{2}}. \tag{5}$$

The restrictions on the $\alpha_i$ follow from the condition that the wavelet coefficient sequence $c_{ij}$ belong to $l^2(\mathbf{Z})$. This is a simple example of a "discrete fractal-wavelet transform" to which, for simplicity, we shall refer as an *IFS on wavelet coefficients* or IFSW.

The goal is to find an IFSM operator $T$ (as defined by Eq. (2)) acting in "physical space", i.e. on functions supported on $\mathbf{R}$, which corresponds to an IFSW operator $M$ acting on the wavelet coefficients. The dilation/translation relations within the wavelet basis provide the key to this goal. Let $w_1(x) = x/2$ and $w_2(x) = x/2 + 1/2$. Then

$$\sqrt{2}B_{00} \circ w_1^{-1} = B_{10} \quad \text{and} \quad \sqrt{2}B_{00} \circ w_2^{-1} = B_{11} \tag{6}$$

since

$$\sqrt{2}\psi_{i,j}\left(w_1^{-1}(x)\right) = \psi_{i+1,j}(x) \quad \text{and} \quad \sqrt{2}\psi_{i,j}\left(w_2^{-1}(x)\right) = \psi_{i+1,j+2^i}(x). \tag{7}$$

Thus, this simple IFSW operator will correspond to the following two-map IFSM with condensation function $\psi(x)$:

$$T(f)(x) = c_{00}\psi(x) + \sqrt{2}\alpha_0 f\left(w_1^{-1}(x)\right) + \sqrt{2}\alpha_1 f\left(w_2^{-1}(x)\right). \tag{8}$$

*Note that the IFSM operator depends on the particular wavelet basis chosen.*

In the simple case of Haar wavelets, the mother wavelet $\psi(x)$ decomposes into nonoverlapping components:

$$\psi(x) = I_{[0,1/2)}(x) - I_{[1/2,1)}(x). \tag{9}$$

As such, the IFSM operator $T$ in Eq. (8) corresponds to a simple two-map IFSM with IFS maps $w_1$ and $w_2$ and grey-level maps $\phi_1(t) = \sqrt{2}\alpha_1 t + 1$ and $\phi_2(t) = \sqrt{2}\alpha_2 t - 1$. If $T$ is contractive, then its fixed point attractor function $\bar{u}$ has [0,1] as support.

However, in the case of other compactly supported wavelets, no such spatial decomposition into separate grey-level maps is possible. As well, the support of the attractor function $\bar{u}$ is necessarily larger than [0,1]. To illustrate, consider the particular IFSW in which $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$. The IFSM operator $T$ in Eq. (8) is contractive. Figures 1(a) and 1(b) show the IFSM attractor functions for, respectively, the Haar wavelet and "Coifman-6" cases. In both cases, we have chosen $b_{00} = 0$ and $c_{00} = 1$.

**Example 2:** Consider the fractal wavelet transform with four block maps as follows:

$$W_1 : B_{10} \to B_{20}, \quad W_2 : B_{11} \to B_{21}, \quad W_3 : B_{10} \to B_{22}, \quad W_4 : B_{11} \to B_{23}, \tag{10}$$

with associated multipliers $\alpha_i$, $1 \leq i \leq 4$. Diagramatically,

$$M \;:\; B_{00} \;\to\; \begin{array}{|c|c|c|c|}\hline \multicolumn{4}{|c|}{c_{00}} \\ \hline \multicolumn{2}{|c|}{c_{10}} & \multicolumn{2}{c|}{c_{11}} \\ \hline \alpha_1 B_{10} & \alpha_2 B_{11} & \alpha_3 B_{10} & \alpha_4 B_{11} \\ \hline \end{array}\;. \tag{11}$$

(The reader may suspect that this has the aroma of a local IFSM.) Now iterate this process, assuming it converges to a limit $\bar{B}_{00}$ which represents the wavelet expansion of a function $\bar{u}$. Then

$$\bar{u} = c_{00}\psi_{00} \;+\; \bar{v}. \tag{12}$$

We need only focus on the function $\bar{v}$ which admits the wavelet expansion

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{2}{|c|}{c_{10}} & \multicolumn{2}{c|}{c_{11}} \\ \hline \alpha_1 B_{10} & \alpha_2 B_{11} & \alpha_3 B_{10} & \alpha_4 B_{11} \\ \hline \end{array}\;. \tag{13}$$

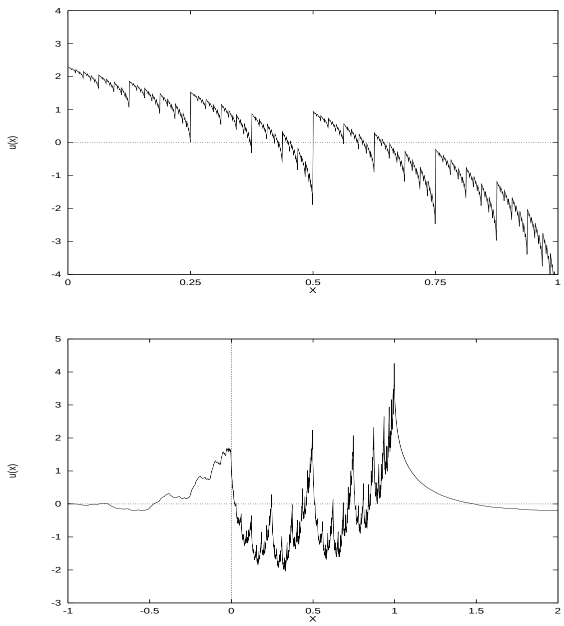


**Figure 1.** Attractor functions $\bar{u}$ for Example 1: (a) Haar wavelet basis, (b) Coifman-6 wavelet basis.

Since $<\psi_{10}, \psi_{11}>= 0$, etc., we may write

$$\bar{v} \;=\; \bar{v}_1 \;+\; \bar{v}_2 \;, \tag{14}$$

where the components $\bar{v}_i$ satisfy the relations

$$\begin{aligned} \bar{v}_1(x) &= c_{10}\psi_{10}(x) \;+\; \alpha_1\sqrt{2}\bar{v}_1(2x) \;+\; \alpha_2\sqrt{2}\bar{v}_2(2x) \\ \bar{v}_2(x) &= c_{11}\psi_{11}(x) \;+\; \alpha_3\sqrt{2}\bar{v}_1(2x-1) \;+\; \alpha_4\sqrt{2}\bar{v}_2(2x-1). \end{aligned} \tag{15}$$

We may consider these equations as defining a kind of *vector IFSM with condensation.* The vector $\bar{v}$ is composed of the orthogonal components $\bar{v}_1$ and $\bar{v}_2$ that satisfy the above fixed point relations in Eq. (15). These equations may be written more compactly as:

$$\bar{v}_i(x) \;=\; b_i(x) \;+\; \sum_{j=1}^{2} \phi_{ij}(\bar{v}_j(w_{ij}^{-1}(x))), \qquad i = 1, 2, \tag{16}$$

where

$$w_{11}(x) \;=\; w_{12}(x) \;=\; \frac{1}{2}x, \;\; w_{21}(x) \;=\; w_{22}(x) \;=\; \frac{1}{2}x \;+\; \frac{1}{2}, \tag{17}$$

$$b_1(x) = c_{10}\psi_{10}(x), \;\; b_2(x) = c_{11}\psi_{11}(x) \;, \tag{18}$$

and

$$\phi_{11}(t) = \alpha_1\sqrt{2}t, \quad \phi_{12}(t) = \alpha_2\sqrt{2}t, \quad \phi_{21}(t) = \alpha_3\sqrt{2}t, \quad \phi_{22}(t) = \alpha_4\sqrt{2}t\ . \tag{19}$$

Note that the contractive IFS maps $w_{ij}$ are mappings from the entire base space $X$ into itself. As such, this IFSM is *not* a local IFSM in general. What appeared to be a "local" transform in wavelet coefficient space is a normal IFSM in the base space. (Again, in the special case of the nonoverlapping Haar wavelets, the above IFSM may be written as a local IFSM.) The "locality" of the block transform has been passed on to the orthogonal components $\bar{v}_1$ and $\bar{v}_2$ of the function $\bar{v}$. These components may be considered as "nonoverlapping" elements of a vector.

This "vector IFSM" is really nothing more than a recurrent IFSM on $B_{00}$ where we split $B_{00}$ as $B_{10} \oplus B_{11}$ and have the IFSM act "between" the components of this splitting.

**Example 3:** Finally, consider the following modification of the fractal wavelet transform in Example 2:

$$W_1 : B_{10} \to B_{20}, \quad W_2 : B_{11} \to B_{21}, \quad W_3 : B_{11} \to B_{22}, \quad W_4 : B_{10} \to B_{23}, \tag{20}$$

with associated multipliers $\alpha_i$, $1 \leq i \leq 4$. Graphically,

$$M \ : \ B_{00} \ \to \ \begin{array}{|c|c|c|c|} \hline \multicolumn{4}{|c|}{c_{00}} \\ \hline \multicolumn{2}{|c|}{c_{10}} & \multicolumn{2}{c|}{c_{11}} \\ \hline \alpha_1 B_{10} & \alpha_2 B_{11} & \alpha_3 B_{11} & \alpha_4 B_{10} \\ \hline \end{array}\ . \tag{21}$$

As above, iterate this process, assuming it converges to a limit $\bar{B}_{00}$ which represents the wavelet expansion of a function $\bar{u}$. Then

$$\bar{u} = c_{00}\psi_{00} \ + \ \bar{v}. \tag{22}$$

where $\bar{v}$ admits the wavelet expansion

$$\begin{array}{|c|c|c|c|} \hline \multicolumn{2}{|c|}{c_{10}} & \multicolumn{2}{c|}{c_{11}} \\ \hline \alpha_1 B_{10} & \alpha_2 B_{11} & \alpha_3 B_{11} & \alpha_4 B_{10} \\ \hline \end{array}\ . \tag{23}$$

As before, we write

$$\bar{v} \ = \ \bar{v}_1 \ + \ \bar{v}_2\ . \tag{24}$$

and determine the relations satisfied by the components $\bar{v}_i$. It is necessary to determine the affine changes of variables involved in mapping $B_{10}$ to $B_{23}$ and $B_{11}$ to $B_{22}$. The result is

$$\begin{aligned} \bar{v}_1(x) &= c_{10}\psi_{10}(x) \ + \ \alpha_1\sqrt{2}\bar{v}_1(2x) \ + \ \alpha_2\sqrt{2}\bar{v}_2(2x) \\ \bar{v}_2(x) &= c_{11}\psi_{11}(x) \ + \ \alpha_3\sqrt{2}\bar{v}_2(2x - \frac{1}{2}) \ + \ \alpha_4\sqrt{2}\bar{v}_1(2x - \frac{3}{2}). \end{aligned} \tag{25}$$

This is a vector or recurrent IFSM with condensation functions which could be written in the compact form of Eq. (16).

# 3 IFSW for Periodic Wavelets on $[0,1]$

In many applications one needs wavelet bases adapted to a compact interval. For this purpose, periodizing the wavelet basis functions is useful. Furthermore, in practical computations, one often deals with finite data sets and periodizing the discrete transform is simple and clean to implement.

For a function $f \in L^2(\mathbb{R})$, define the periodized version of $f$ to be

$$f^*(x) = \sum_i f(x-i) \tag{26}$$

where the sum is over all integer $i$. This process "wraps" the function $f$ around over the interval $[0,1]$ and sums the various contributions.

Let $\phi$ be a scaling function for an MRA on $\mathbb{R}$ and $\psi$ be the associated "mother" wavelet. Then the periodized wavelets $\psi^*_{ij}(x)$ form an orthonormal basis of $L^2[0,1]$ and we have a nested MRA structure as in the case of wavelets on $\mathbb{R}$ [Dau]. Unfortunately, the nice translation/scaling relations (represented by Eq. (7) ) no longer hold for these periodized wavelets. To see this, consider the following simple example. By definition,

$$\begin{aligned} \psi^*_{1j}(x) &= \sum_k \psi_{1j}(x-k) \\ &= \sum_k \psi(2x-2k-j), \end{aligned} \tag{27}$$

where the latter equation follows from the scaling property of the wavelets. However, the function

$$\begin{aligned} \psi^*(2x-j) &= \sum_k \psi(2x-j-k) \\ &\neq \psi^*_{1j}(x). \end{aligned} \tag{28}$$

As a result, the rather simple scaling analysis of Section 2 is not applicable here. The IFSM associated with an IFSW will have to be different in the periodic case than in the case of functions on $\mathbb{R}$.

Suppose that we are given an IFS on the wavelet coefficient tree for some periodized wavelets on the interval $[0,1]$. This operator induces an operator

$$T : L^2[0,1] \to L^2[0,1].$$

Since we are interested in periodic functions, we think of the unit circle with $[0,1]$ as the "natural" fundamental domain for the periodization.

Let $\psi$ be the mother wavelet for some wavelet decomposition. Suppose that the support of $\psi$ lies in the interval $[0,N]$. Then

$$supp(\psi^j) \equiv supp(\psi(2^n x - j)) \subset 2^{-n}[j, j+N] = [\frac{j}{2^n}, \frac{j+N}{2^n}]$$

with $j$ running from 0 to $2^n - 1$. So, for sufficiently large $n$ we have that their supports have length less than 1 and that all the supports lie in the interval $[0, 3/2]$.

We now define the IFS maps $w_1, w_2, \hat{w}_1$ and $\hat{w}_2$ on the circle. These maps will be induced by corresponding maps $W_1(x) = x/2$ and $W_2(x) = x/2 + 1/2$ on $\mathbb{R}$. For the maps $w_1$ and $w_2$, we use the coordinate system on the circle induced by the fundamental domain $[0, 1]$. Notice that the image of $[0, 1]$ under $w_1$ is $[0, 1/2]$ – this map would not be a continuous map on the circle, so we must make a "cut" at the point $x = 0 = 1$. Thus, these maps are, in some sense, only "local" maps. For the maps $\hat{w}_1$ and $\hat{w}_2$, we use the coordinate system on the circle induced by the domain $[1/2, 3/2]$. Here the "cut" point is $x = 1/2$ (the point antipodal to the point 0). Thus, the image of the circle under $\hat{w}_1$ is $[1/4, 3/4]$ – the "middle" of the circle – and under $\hat{w}_2$ is $[3/4, 5/4]$. Again, these are "local" maps.

With these definitions, the usual scaling and translation relations among the wavelets are modified as follows. For $i \geq n$,

$$\sqrt{2}\psi^*_{i,j}(w_1^{-1}(x)) = \psi^*_{i+1,j}(x) \quad \text{and} \quad \sqrt{2}\psi^*_{i,j}(w_2^{-1}(x)) = \psi^*_{i+1,2^i+j}(x) \tag{29}$$

for $0 \leq j < 2^{i-1}$ and

$$\sqrt{2}\psi^*_{i,j}(\hat{w}_1{}^{-1}(x)) = \psi^*_{i+1,j}(x) \quad \text{and} \quad \sqrt{2}\psi^*_{i,j}(\hat{w}_2{}^{-1}(x)) = \psi^*_{i+1,2^i+j}(x) \tag{30}$$

for $j \geq 2^{i-1}$.

The functions $\psi^*_{i,j}$ are supported on the interval $[0, 1]$ for $j < 2^{i-1}$ and on the interval $[1/2, 3/2]$ for $j \geq 2^{i-1}$. Thus, we use the $w_i$'s for the first ones and the $\hat{w}_i$'s for the second ones. In our block notation, these scaling and translation relations become

$$\sqrt{2}B_{i,j} \circ w_1^{-1} = B_{i+1,j} \quad \text{and} \quad \sqrt{2}B_{i,j} \circ w_2^{-1} = B_{i+1,2^i+j} \tag{31}$$

for $0 \leq j < 2^{i-1}$ and

$$\sqrt{2}B_{i,j} \circ \hat{w}_1{}^{-1} = B_{i+1,j} \quad \text{and} \quad \sqrt{2}B_{i,j} \circ \hat{w}_2{}^{-1} = B_{i+1,2^i+j} \tag{32}$$

for $j \geq 2^{i-1}$.

These relations are the crucial observation needed to show that the IFS on the wavelet coefficients induces an IFSM operator on $L^2[0, 1]$ (in the usual sense of an IFSM). In order to do this, we break the space $L^2[0, 1]$ into two orthogonal parts. The first part is the space spanned by the first $n$ levels of the wavelet basis:

$$V_1 = span\{\psi^*_{i,j} \mid 0 \leq i < n, 0 \leq j < 2^i\}$$

and the second part is the orthogonal complement of $V_1$ in $L^2[0, 1]$ so that

$$V_1 \oplus V_2 = L^2[0, 1].$$

Notice that $T$ maps $V_1$ into both $V_1$ and $V_2$. However, $T(V_2) \subset V_2$. Thus, we have a sort of "recurrent" IFSM between $V_1$ and $V_2$. We represent the operator $T$ as

$$T = (T_1 + T_2) \oplus T_3 \tag{33}$$

with

$$T_1 : V_1 \to V_1 \quad \text{and} \quad T_2 : V_1 \to V_2 \quad \text{and} \quad T_3 : V_2 \to V_2 \tag{34}$$

Let $v$ be the fixed point of the operator $T$, thus $T(v) = v$. Writing $v = v_1 \oplus v_2$, we obtain

$$T_1(v_1) \oplus (T_2(v_1) + T_3(v_2)) = v_1 \oplus v_2 \tag{35}$$

and thus

$$\begin{aligned} T_1(v_1) &= v_1 \qquad (36) \\ T_2(v_1) + T_3(v_2) &= v_2. \qquad (37) \end{aligned}$$

So the IFS on $V_1$ is a simple IFS and the IFS on $V_2$ is an IFS with condensation function $T_2(v_1)$.

The vector $v_1$ is relatively easy to calculate since $V_1$ is finite dimensional. One needs only to compute the coefficients of $v_1$ with respect to the basis $\psi^*_{i,j}$ for appropriate $i, j$. These coefficients are given by the IFSW operator. From $v_1$ one obtains the function $T_2(v_1)$ – the condensation function for the IFSM on $V_2$. The IFSM on $V_2$ basically corresponds to the case of non-periodized wavelets (i.e. wavelets on $\mathbb{R}$).

Finally, we return to the fractal-wavelet transform of Example 1 applied to coefficient trees for periodized wavelets. In the Haar case, the result is trivially the same as in Figure 1(a). The periodic attractor function $\bar{u}$ for the Coifman-6 periodized wavelets is shown in Figure 2. (Note that the wavelet coefficients $c_{ij}$ for all three graphs are identical.)

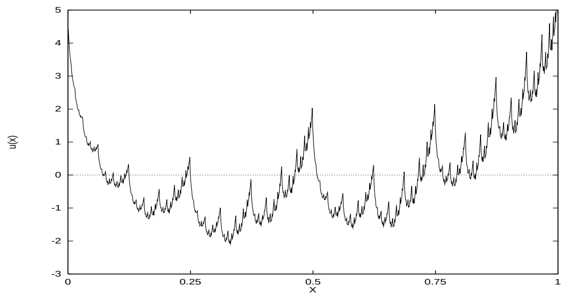


**Figure 2.** Attractor function $\bar{u}$ for IFSW in Example 1 using Coifman-6 periodized wavelet functions.

## 4 Fractal Dimension of IFSM and IFSW

In this section we discuss the dimension of the attractor of an IFSM or IFSW (which we think of as an IFSM with condensation). We will primarily be interested in the box counting dimension, although in many cases this is the same as the Hausdorff dimension. Our methods are direct adaptations of the methods in [B, F], thus we give only the sketches of the proofs.

Recall that our usual IFSM operator has the form

$$T(f)(x) = \sum_i \alpha_i f(w_i^{-1}(x)) + \beta_i \chi_i, \tag{38}$$

where $\chi_i$ is the characteristic function of $X_i = w_i([0,1])$ and $w_i(x) = s_i x + b_i$.

We can think of the piecewise constant parts $\beta_i \chi_i$ as being a piecewise-defined "condensation function" for the IFSM. Thus, the more general form of the IFSM is given by

$$T(f)(x) = \sum_i \alpha_i f(w_i^{-1}(x)) + \theta(x) \tag{39}$$

We obtain the following theorem on the fractal dimension of the graph of the attractor for the IFSM (39). Notice that if $\theta$ is $C^1$, then the graph of $\theta$ will have dimension 1.

**Theorem 1** *Suppose that the IFSM operator $T$ as defined in equation (39) is non-overlapping and that the dimension of the graph of $\theta(x)$ is 1. Then the dimension of the fixed point of $T$ is the unique solution $D$ to*

$$\sum_i |\alpha_i| |s_i|^{D-1} = 1 \tag{40}$$

*if $\sum_i |\alpha_i| > 1$ and 1 otherwise.*

**Proof:** As in ([Bar], p. 226), we give just a heuristic reasoning.

Denote by $N(\epsilon)$ the number of boxes of side length $\epsilon$ that intersect the graph of the fixed point function. Then by the self-similarity,

$$N(\epsilon) \approx \sum_i |\frac{\alpha_i}{s_i}| N(\frac{\epsilon}{|s_i|}) + C\epsilon^{-1}$$

where the last term appears because of the "condensation function" $\theta$. Notice that if we substitute the ansatz $N(\epsilon) = C\epsilon^{-D}$ into this equation, we obtain

$$1 = \sum_i |\alpha_i| |s_i|^{D-1}$$

which is the desired relation (the $C\epsilon^{-1}$ term drops out as $\epsilon \to 0$). ∎

The following Corollary is also a special case of results in [B, F] (the special case where there is no shear in the IFS maps).

**Corollary 2** *For the affine IFSM operator $T$ as defined in equation (38), the dimension of the fixed point of $T$ is the unique solution $D$ to equation (40) if $\sum_i |\alpha_i| > 1$ and 1 otherwise.*

Now, in the case of an IFSM corresponding to an IFSW, the mappings are certainly not non-overlapping. Thus, the previous results do not apply. As in the case of a usual IFS, we obtain an upper bound on the dimension.

**Theorem 3** *Suppose that $\psi$, the "mother wavelet" for the wavelet MRA, is differentiable. Then for the affine IFSW operator $S$ defined as in equation (5) the dimension of the fixed point of $S$ is bounded above by the unique solution $D$ to*

$$\sum_i |\alpha_i||s_i|^{D-1} = 1 \tag{41}$$

*if $\sum_i |\alpha_i| > 1$ and 1 otherwise.*

**Proof:** Let $N(f, \epsilon)$ be the number of boxes in a "column" that intersect the graph of $f$. Then

$$N(f+g, \epsilon) \leq N(f, \epsilon) + N(g, \epsilon). \tag{42}$$

Using this observation one can obtain the upper bound in the case where the images in the IFSM are overlapping in a similar manner as one obtains the upper bound for the similarity dimension for geometric IFS. ▮

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