Lecture 7

Equilibrium or "steady-state" temperature distributions (cont'd)

Equilibrium temperature distributions for other cases

We may investigate the existence of steady state distributions for other situations, including:

1. **Mixed boundary conditions:** For example $u(0) = T_1$, $u'(L) = 0$. This would correspond to a heat bath in contact with the rod at $x = 0$ and an insulated end at $x = L$. Once again, the steady-state solution would assume the form

$$u_{eq}(x) = C_1 x + C_2. \tag{1}$$

The condition $u(0) = T_1$ implies that $C_2 = T_1$. The second condition $u'(L) = 0$ implies that $C_1 = 0$. Therefore, there is a unique steady-state solution:

$$u_{eq}(x) = T_1. \tag{2}$$

2. **The presence of heat sources:** In this case, we look for steady-state solutions to the following problem (assuming constant coefficients):

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q(x), \tag{3}$$

where, recall, $Q(x)$ represents the rate of thermal energy production (or depletion) at x. A steady state solution $u(x,t) = u_{eq}(x)$ would have to satisfy the ODE,

$$\frac{d^2u}{dx^2} = -\frac{Q(x)}{K_0}. \tag{4}$$

We antidifferentiate twice to give the general solution,

$$u(x) = -\frac{1}{K_0} \int^x \int^t Q(s)\, ds\, dt + C_1 x + C_2. \tag{5}$$

One would then impose the boundary conditions relevant to the problem.

Example 1: Suppose that $Q(x) = x$, $0 \le x \le L$. For simplicity, we assume that $K_0 = 1$. Then the ODE in (4) becomes

$$\frac{d^2u}{dx^2} = -\frac{1}{K_0} x = -x. \tag{6}$$

Antidifferentiating twice produces the general solution

$$u(x) = -\frac{1}{6}x^3 + C_1 x + C_2. \tag{7}$$

Suppose that we imposed the boundary conditions

$$u(0) = 0, \quad u(L) = 0, \tag{8}$$

i.e., fixed temperatures at the two ends. The condition $u(0) = 0$ implies that $C_2 = 0$. The condition $u(L) = 0$ implies that

$$-\frac{1}{6}L^3 + C_1 L = 0, \tag{9}$$

or $C_1 = \frac{L^2}{6}$. Thus the steady-state temperature is

$$u_{eq}(x) = \frac{1}{6}(L^2 x - x^3). \tag{10}$$

Note that this steady-state distribution is *not* a constant temperature distribution, because of the presence of sources. Nevertheless, the boundary conditions dictate that the steady-state temperature is zero at the endpoints. Therefore, as x increases from zero, the temperature increases, reaches a maximum, then decreases to zero as $x \to L$. Since $Q(x) = x$, the rod is being heated at a greater rate as x increases from 0 to L. This is why the graph of the steady-state temperature function, shown below for the case $L = 1$, is somewhat skewed to the right, dropping more quickly to zero as $x \to L$ than it does as $x \to 0$.

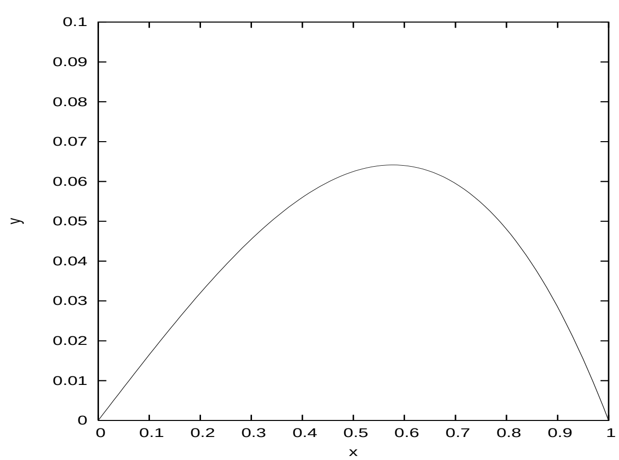

Figure 1: Steady state distribution $u_{eq}(x) = \frac{1}{6}(x - x^3)$ for example in text.

But this graph tells us even more. Recall that, in general, the (rightward) flux at a point is given by

$$\phi(x,t) = -K_0 \frac{\partial u(x,t)}{\partial x}. \tag{11}$$

For the steady state distribution $u_{eq}(x)$ shown above, where $L = 1$ and $K_0 = 1$, we compute the the fluxes at the endpoints to be

$$\phi(0) = -u'_{eq}(0) = -\frac{1}{6}, \qquad \phi(1) = -u'_{eq}(1) = \frac{1}{3}. \tag{12}$$

This shows that heat is flowing to the left $x = 0$ and to the right at $x = 1$. This has to be the case for this steady state distribution, because there are internal sources of heat, given by the rate function $Q(x) = x$. At steady state, we expect that the net rate of heat flow through the ends of the rods is equal to the rate of heat production in the rod. Let us verify this below.

The total rate of heat production in the rod is given by (we assume that the cross-sectional area is $A = 1$)

$$\int_0^1 Q(x)\,dx = \int_0^1 x\,dx = \frac{1}{2} \tag{13}$$

The rate of flow of heat through the ends is therefore given by

$$\phi(0) - \phi(1) = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}. \tag{14}$$

The negative sign indicates the the net flow is *outward*. And the rate of outward flow of heat, i.e., $\frac{1}{2}$, is seen to be equal to the rate of heat production in the rod, as would be expected in a steady-state situation: There is no buildup or depletion of heat in time.

The following problem was asked by a student in class, which was greatly appreciated, since it gives some further insight into these problems.

Example 2: Suppose that $Q(x) = x$, $0 \le x \le L$, and $K_0 = 1$, as in Example 1, but the ends are *insulated*, i.e., there are no-flux conditions at the endpoints, i.e.,

$$u'(0) = 0, \quad u'(L) = 0. \tag{15}$$

Before we proceed, we would expect, on physical grounds, that if no heat is allowed to escape through the ends of the rod, and heat is continually being generated along the rod, then the total thermal energy of the rod will increase in time without bound. Therefore, it be impossible

possible for an equilibrium, time-independent, solution to exist. We'll see that this is confirmed by the mathematics.

Recall that the general form of the equilibrium solution $u_{eq}(x)$ is given by

$$u(x) = -\frac{1}{6}x^3 + C_1 x + C_2. \tag{16}$$

In order to impose the boundary conditions, we first compute

$$u'(x) = -\frac{1}{3}x^2 + C_1. \tag{17}$$

The first condition, $u'(0) = 0$, implies that $C_1 = 0$, i.e.,

$$u'(x) = -\frac{1}{3}x^2. \tag{18}$$

Imposing the second condition, $u'(L) = 0$, implies that $L = 0$, which is not acceptable. Therefore, no equilibrium temperature distribution exists.

We now turn to the problem of actually solving the heat equation!

Linear operators and superposition of solutions

`Sections 2.1-2.2 of text by Haberman`

We now return to the 1D heat equation with source term

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}. \tag{19}$$

The boundary conditions and initial condition are not important at this time. We also consider the associated *homogeneous* form of this equation, correponding to an absence of any heat sources, i.e.,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{20}$$

The following discussion will most probably bring back memories of ideas that you saw in a previous course on differential equations, e.g., AMATH 351.

It is not too difficult to show that if $u_1(x,t)$ and $u_2(x,t)$ are solutions to the homogeneous equation (20), then any linear combination of these solutions, i.e.,

$$u(x,t) = c_1u_1(x,t) + c_2u_2(x,t), \qquad c_1, c_2 \in \mathbf{R}, \tag{21}$$

is also a solution to (20):

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}[c_1u_1 + c_2u_2] \\
&= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} \\
&= c_1k\frac{\partial^2 u_1}{\partial x^2} + c_2k\frac{\partial^2 u_2}{\partial x^2} \qquad \text{(from Eq. (20))} \\
&= k\frac{\partial^2}{\partial x^2}[c_1u_1 + c_2u_2] \\
&= k\frac{\partial^2 u}{\partial x^2}.
\end{aligned} \tag{22}$$

It is also not difficult to show that this property is *not* shared by the inhomogeneous PDE in (19): If $v_1(x,t)$ and $v_2(x,t)$ are solutions of (19), then it is not true that *any* linear combination of these solutions, i.e.,

$$v(x,t) = v_1u_1(x,t) + v_2u_2(x,t), \qquad c_1, c_2 \in \mathbf{R}, \tag{23}$$

is also a solution to (19). Yes, the above combination may be true for special cases, i.e., $c_2 = 0$, but it is not true for *all* values $c_1, c_2 \in \mathbf{R}$.

Here is a "proof" that was not done in class:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}[c_1 v_1 + c_2 v_2] \\ &= c_1 \frac{\partial v_1}{\partial t} + c_2 \frac{\partial v_2}{\partial t} \\ &= c_1 \left[k \frac{\partial^2 u_1}{\partial x^2} + \frac{Q}{c\rho}\right] + c_2 \left[k \frac{\partial^2 u_2}{\partial x^2} + \frac{Q}{c\rho}\right] \\ &= k \frac{\partial^2}{\partial x^2}[c_1 v_1 + c_2 v_2] + (c_1 + c_2)\frac{Q}{c\rho} \\ &= k \frac{\partial^2 v}{\partial x^2} + (c_1 + c_2)\frac{Q}{c\rho}. \end{aligned} \tag{24}$$

We see that only in the special case $c_1 + c_2 = 1$ does v satisfy (19). For a "superposition principle" to be valid, it must hold for all values of the constants c_1 and c_2. Therefore the superposition principle is not valid for the inhomogeneous heat equation.

The fact that linear superpositions of solutions to the homogeneous equation (20) are also solutions is made possible by the *linearity* of the derivative operators that make up the PDE. Eqs. (19) and (20) can be written in the form

$$L(u) = g, \tag{25}$$

where

$$L = \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}, \qquad g = \frac{Q}{c\rho}. \tag{26}$$

L is a linear operator, i.e.,

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2). \tag{27}$$

Eq. (25) is a linear equation in u:

1. If g is nonzero, as in Eq. (20), then the linear equation is *inhomogeneous*.

2. If g is zero, as in Eq. (19), then the linear equation is *homogeneous*.

Linear homogeneous equations admit *superposition of solutions*: If u_1 and u_2 are solutions of $L(u) = 0$, then the linear combination

$$u = c_1 u_1 + c_2 u_2 \tag{28}$$

is also a solution of $L(u) = 0$.

Proof:

$$\begin{aligned} L(c_1u_1 + c_2u_2) &= c_1L(u_1) + c_2L(u_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0. \end{aligned} \tag{29}$$

Once again, this property applies to the 1D heat equation, Eq. (19), with no sources, i.e., $Q = 0$.

The superposition property also applies to *linear "homogeneous" boundary conditions*, as discussed in the text. Very briefly, suppose that we have the boundary conditions

$$u(0) = 0, \qquad u(L) = 0. \tag{30}$$

They are "homogeneous" because the right hand sides are zero. Now suppose that u_1 and u_2 satisfy these boundary conditions. Let

$$v = c_1u_1 + c_2u_2, \qquad \text{for any} \quad c_1, c_2 \in \mathbf{R}. \tag{31}$$

Then

$$\begin{aligned} v(0) &= c_1u_1(0) + c_2u_2(0) = c_1 \cdot 0 + c_2 \cdot 0 = 0 \\ v(L) &= c_1u_1(L) + c_2u_2(L) = c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned} \tag{32}$$

implying that v satisfies (30). The same result applies to the more general linear homogeneous boundary conditions of the form

$$\alpha_1u(0) + \alpha_2u'(0) = 0, \qquad \beta_1u(L) + \beta_2u'(L) = 0. \tag{33}$$

The method of "Separation of Variables"

We now describe the *separation of variables* method, one of the first "tools" to be tried in the solution of PDEs.

As an example, we consider 1D heat equation without sources,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{34}$$

with zero-endpoint boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \tag{35}$$

and initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L. \tag{36}$$

In the separation of variables method, we assume a solution of the form

$$u(x,t) = \phi(x)G(t). \tag{37}$$

(This notation is not optimal, given that "ϕ" was also used to represent the flux at a point. However, it is the notation used in the text, and we use it here in order to maintain some consistency.) The boundary conditions will have to be satisfied by the ϕ functions, i.e.,

$$\phi(0) = 0, \qquad \phi(L) = 0. \tag{38}$$

We substitute (37) into (34):

$$\frac{\partial}{\partial t}[\phi(x)G(t)] = k\frac{\partial^2}{\partial x^2}[\phi(x)G(t)] \tag{39}$$

implying

$$\phi(x)G'(t) = k\phi''(x)G(t). \tag{40}$$

We "separate the variables" into t-dependent terms on the left and x-dependent terms on the right:

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)}. \tag{41}$$

We include the k on the left for reasons that will be clear later.

Keeping in mind that x and t are independent variables, the only way that the above equality can hold is if both terms are constant. We write

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = \mu, \tag{42}$$

where $\mu \in \mathbf{R}$ is known as the *separation constant*.

Let us now examine the time-dependent equation

$$\frac{G'(t)}{kG(t)} = \mu, \tag{43}$$

which we rewrite as

$$G'(t) - \mu k G(t) = 0, \tag{44}$$

a first-order ODE in $G(t)$ with general solution

$$G(t) = Ce^{\mu k t}. \tag{45}$$

This implies that the solution $u(x,t)$ to the heat equation has the form

$$u(x,t) = C\phi(x)e^{\mu k t}. \tag{46}$$

We can argue, on physical grounds, that the separation constant μ should be *negative*. Recall that the steady-state temperature distribution for the heat equation with zero endpoint conditions is $u_{eq}(x) = 0$. Furthermore, we expect all solutions $u(x,t)$ to approach this solution as $t \to \infty$. The only way that this could happen is if μ is negative. In the next lecture, however, we'll prove this result.

Lecture 8

The "Separation of Variables" method (cont'd)

We now examine the spatial-dependent equation in ϕ,

$$\frac{\phi''(x)}{\phi(x)} = \mu, \tag{47}$$

which may be rearranged to give

$$\phi''(x) - \mu\phi(x) = 0, \qquad \phi(0) = 0, \quad \phi(L) = 0. \tag{48}$$

This is a second order, linear *boundary value problem* in the unknown function $\phi(x)$. You may be familiar with such problems from AMATH 351. The goal is to determine ϕ as well as the separation constant μ.

1. **Case 1:** $\mu > 0$**.** Once again, the BVP is given by

   $$\phi'' - \mu\phi = 0, \qquad \phi(0) = \phi(L) = 0. \tag{49}$$

   The general solution to the DE is

   $$\phi(x) = C_1 e^{\sqrt{\mu}x} + C_2 e^{-\sqrt{\mu}x} \tag{50}$$

   The condition $\phi(0) = 0$ implies that $C_1 + C_2 = 0$, or $C_2 = -C_1$, so that

   $$\phi(x) = C_1[e^{\sqrt{\mu}x} - e^{-\sqrt{\mu}x}] = 2C_1 \sinh(\sqrt{\mu}x). \tag{51}$$

   The condition $\phi(L) = 0$, implies that $C_2 \sinh(\sqrt{\mu}L) = 0$, which is satisfied only if $C_2 = 0$. This follows from the fact that $\sinh(x)$ is zero only at $x = 0$. The net result: The only solution to the BVP for this case is the trivial solution $\phi(x) = 0$.

2. **Case 2:** $\mu = 0$**.** The BVP becomes

   $$\phi''(x) = 0, \qquad \phi(0) = \phi(L) = 0. \tag{52}$$

   The general solution to the DE is

   $$\phi(x) = C_1 x + C_2. \tag{53}$$

   The condition $\phi(0) = 0$ implies that $C_2 = 0$. The condition $\phi(L) = 0$ implies that $C_1 L = 0$, or $C_1 = 0$. Thus the only solution to the BVP is the trivial solution $\phi(x) = 0$.

3. **Case 3:** $\mu < 0$. For convenience, we shall set $\lambda = -\mu$ so that $\lambda > 0$. The BVP in (48) becomes

$$\phi''(x) + \lambda\phi(x) = 0, \qquad \phi(0) = 0, \quad \phi(L) = 0. \tag{54}$$

The general solution to the homogeneous DE is

$$\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x). \tag{55}$$

We now apply the boundary conditions to this solution. Firstly

$$\phi(0) = 0 \rightarrow C_1 + C_2 \cdot 0 = 0 \rightarrow C_1 = 0. \tag{56}$$

Therefore $\phi(x) = C_2 \sin(\sqrt{\lambda}x)$. The condition $\phi(L) = 0$ implies that

$$\sin(\sqrt{\lambda}L) = 0. \tag{57}$$

Since L is fixed, we must adjust λ in order that the above equality is satisfied:

$$\sqrt{\lambda}L = n\pi, \qquad n = 1, 2, 3, \cdots. \tag{58}$$

This implies that λ can assume only discrete values, which we denote as

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \cdots. \tag{59}$$

The solutions $\phi_n(x)$ associated with these λ-values are given by

$$\phi_n(x) = C_2 \sin(\sqrt{\lambda_n}x) = C_2 \sin\left(\frac{n\pi x}{L}\right). \tag{60}$$

The λ_n are referred to as *eigenvalues* and the ϕ_n as *eigenfunctions*.

This comes from rewriting the boundary value problem for ϕ as

$$L\phi = \lambda\phi, \qquad L = -\frac{d}{dx^2}. \tag{61}$$

More on this later.

Here we mention that the *discretization* or *quantization* of the parameter λ is due to the *boundary conditions* imposed on the solutions. This is a very common occurence in applied mathematics, science and engineering, e.g., resonant vibrational frequencies in mechanics, acoustics, electromagnetism or discrete energy levels in quantum mechanics.

Let us now return to the original separated equations in t and x, acknowledging the discretization of the separation constant $-\lambda$:

$$\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda_n. \tag{62}$$

This implies that for each eigenvalue λ_n there will be an associated spatial function $\phi_n(x)$ and a time-dependent function (omitting the constant)

$$G_n(t) = e^{-\lambda_n kt} = e^{-k(n\pi/L)^2 t}. \tag{63}$$

We now combine these two functions to produce a set of separation-of-variable solutions to the heat equation with zero-endpoint boundary conditions:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}, \qquad n = 1, 2, \cdots. \tag{64}$$

By construction, each of these functions is a solution to the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{65}$$

with zero-endpoint boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \tag{66}$$

However, at time $t = 0$,

$$u_n(x,0) = \sin\left(\frac{n\pi x}{L}\right). \tag{67}$$

In general, therefore, these functions do *not* satisfy the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{68}$$

where $f(x)$ is a prescribed initial temperature distribution function.

Fortunately, however, we have the following important results:

1. Any linear combination of the $u_n(x,t)$, i.e.,

$$u(x,t) = \sum_{n=1}^{M} c_n u_n(x,t) = \sum_{n=1}^{M} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}, \tag{69}$$

   is also a solution to the above heat equation that satisfies the boundary conditions. Once again, it will not necessarily satisfy the initial condition. The initial value associated with this solution is

$$u(x,0) = \sum_{n=1}^{M} c_n \sin\left(\frac{n\pi x}{L}\right). \tag{70}$$

2. Any "square integrable" function $f(x)$ on $[0, L]$ can be approximated *arbitrarily closely* by a linear combination of the functions $\phi_n(x)$, i.e.,

$$f(x) \approx \sum_{n=1}^{M} a_n \phi_n(x) = \sum_{n=1}^{M} a_n \sin\left(\frac{n\pi x}{L}\right). \tag{71}$$

As M increases, the above approximation gets "better", i.e., the magnitude of the "error" in approximating $f(x)$ by the M-term sum decreases. Of course, we have not defined the "error" – more on this later.

In the limit $M \to \infty$, the "error" approaches zero, so that we may write

$$\text{“}f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).\text{”} \tag{72}$$

This is the basis of "Fourier series" or "Fourier approximation." We'll simply state a couple of results below and return to them later in the course.

Let us first note the orthogonality relations satisfied by the functions $\phi_n(x)$:

$$\int_0^L \phi_m(x)\phi_n(x)\, dx = \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n \end{cases} \tag{73}$$

We say that the $\phi_n(x)$ form an *orthonormal set* of functions on $[0, L]$. (Note that the functions are *not* orthonormal.)

Furthermore, these functions form a *complete basis set* for the space of "square-integrable" functions $L^2[0, L]$. In general, the space of square-integrable functions on an interval $[a, b]$, denoted as $L^2[a, b]$, is the set of functions $f : [a, b] \to \mathbf{R}$ for which

$$\int_a^b f(x)^2\, dx < \infty. \tag{74}$$

(The above definition includes the special cases $a = -\infty$ or $b = \infty$ or both.)

This means that for every function $f \in L^2[0, L]$, there exists a unique expansion of the form in Eq. (72). The coefficients a_n are known as the *expansion coefficients* or *Fourier coefficients* of $f(x)$. The space $L^2[0, L]$ includes all continuous functions on $[0, L]$, continuously differentiable functions on $[0, L]$, piecewise continuous functions, and many more classes of functions. Let it suffice for now to say that this space is certainly more than we need in our applications. More on this later.

Lecture 9

The question now remains: Given a function $f(x)$ on $[0, L]$, how do we determine the coefficients a_n in the expansion

$$f(x) = \sum_n^{\infty} a_n \phi_n(x) = \sum_n^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\ dx \quad ? \tag{75}$$

You most probably saw the answer to this question in Calculus 4 or equivalent. We'll derive it quickly below.

First, select an integer $k \geq 1$ and multiply the left and rightmost sides of the above equation by $\phi_k(x) = \sin\left(\frac{k\pi x}{L}\right)$. Then integrate both sides w.r.t. x from 0 to L:

$$\begin{aligned} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right)\ dx &= \int_0^L \sum_n^{\infty} a_n \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\ dx \\ &= \sum_n^{\infty} \int_0^L a_n \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\ dx \\ &= a_k \left(\frac{L}{2}\right). \end{aligned} \tag{76}$$

(In the above, we have bypassed all mathematical rigour. The procedures of integration and infinite summation both involve limiting processes, which would have to be taken into account. As well, the inversion of the order of summation and integration would also have to be justified. This is done by working with finite partial sums and then taking limits.) The final result follows from the orthogonality of the ϕ_n, cf. Eq. (73) of the previous lecture. A rearrangement produces the desired result

$$a_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right). \tag{77}$$

Note: This result, which you most probably recall from Calculus 4, states that the coefficient a_k is obtained from the *scalar product* of the functions $f(x)$ and $\phi_k(x)$. It is an "infinite-dimensional" analogue of the result involving vectors in $\mathbf{R}^n$. Suppose that $g_1, g_2, \cdots g_n$ form an orthogonal set of vectors in $\mathbf{R}^n$, i.e., $g_i \cdot g_j = 0$ if $i \neq j$ and $g_i \cdot g_i = \|g_i\|^2$. Now let a be an arbitrary vector in $\mathbf{R}^n$. Then a admits a unique expansion in terms of the g_i:

$$a = c_1 g_1 + c_2 g_2 + \cdots + c_n g_n. \tag{78}$$

For each $k \in \{1, 2, \cdots n\}$, we form the scalar product of g_k with both sides of the above equation:

$$g_k \cdot a = c_1 g_k \cdot g_1 + \cdots + c_n g_k \cdot g_n = c_k g_k \cdot g_k. \tag{79}$$

Thus we have

$$c_k = \frac{a \cdot g_k}{g_k \cdot g_k}, \tag{80}$$

which is the finite-dimensional analogue to Eq. (77). The reader is referred to the section entitled "Appendix to 2.3" on page 58 of the text by Haberman for a more detailed discussion.

We have now arrived at the final solution, as yielded by the method of Separation of Variables, to the heat equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{81}$$

with zero-endpoint (homogeneous) boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \tag{82}$$

and initial condition (in time),

$$u(x,t) = f(x), \qquad 0 \le x \le L. \tag{83}$$

It is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}, \tag{84}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{85}$$

Note that all the time-dependent functions $G_n(t)$ involve exponentials that decay in time, i.e., $G_n(t) \to 0$ as $t \to \infty$. As a consequence, all solutions $u(x,t)$ decay to zero in this limit, i.e.,

$$u(x,t) \to 0 \quad \text{as} \quad t \to \infty, \tag{86}$$

independent of $f(x)$. But recall that the equilibrium solution to this problem, i.e., the time-independent solution $u_{eq}(x)$ corresponding to zero-endpoint boundary conditions, is given by $u_{eq}(x)$. In other words,

$$u(x,t) \to u_{eq}(x) \quad \text{as} \quad t \to \infty, \tag{87}$$

as we claimed earlier.

Example: We consider $f(x) = 100$, the constant temperature distribution on $[0, L]$. The Fourier coefficients a_n are given by

$$\begin{aligned} a_n &= \int_0^L 100 \sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{200}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L \\ &= \frac{200}{n\pi}[1 - \cos(n\pi)] \\ &= \begin{cases} 0, & n \text{ even.} \\ \frac{400}{n\pi}, & n \text{ odd.} \end{cases} \end{aligned} \tag{88}$$

In the two plots on the next page are shown the Fourier approximations to $u(x, 0) = f(x)$ and some "snapshots" of $u(x, t)$, using $M = 100$ and $M = 1000$ terms of the expansion. The values of the parameters are $k = 1$ and $L = 1$. Approximations to $u(x, t_n)$, where $t_n = n\Delta t$, with $\Delta t = 0.05$, for $n = 0, 1, \cdots 10$ are shown. Clearly, the $M = 1000$ case approximates the initial temperature distribution more accurately. This would lead one to have more confidence with the approximations to $u(x, t)$ for $M = 1000$. However, as t increases, we notice that the approximations to $u(x, t_n)$ seem to look more and more alike. This can be explained by the fact that the higher frequency terms in the expansion of $u(x, t)$ decay more rapidly than the lower frequency ones.

Let us examine this in a little more detail. Recall that the expansion for $u(x, t)$ assumes the form

$$u(x, t) = \sum_0^\infty a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}. \tag{89}$$

As n increases, the magnitudes of the terms in the exponentials increase quadratically with n. For sufficiently large t, the first term, i.e., $n = 1$ will provide the major contribution to the above sum. In other words, the first term will provide an excellent approximation to $u(x, t)$. This is discussed in more detail in the text by Haberman.

There is one other noticeable feature of the solutions which may be quite disturbing. Recall that at $t = 0$, the temperature at any point of the rod, including $x = 0$ was assumed to be 100, i.e., nonzero. For any $t > 0$ however small, the solution at $x = 0$ and $x = L$ is zero, i.e., $u(0, t) = u(L, t) = 0$. (After all, the spatial functions $\phi_n(x)$ were designed to satisfy the zero-endpoint boundary conditions.) In other words, the temperature at $x = 0$ jumps discontinuously from a nonzero amount to zero, implying an infinite rate of heat transfer which is not physically true. (After all, it would violate relativity.) This is, in essence, a consequence of the model, in particular, the fact that we used the Fourier "law"

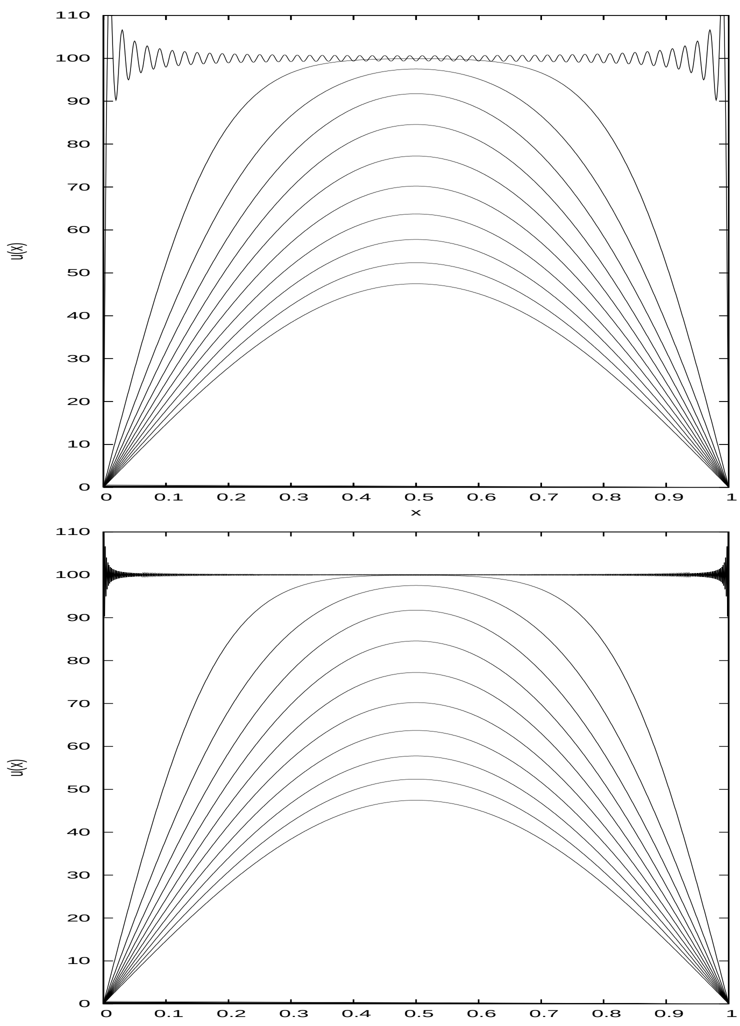

Figure 1: Fourier series approximations to the solution

$$u(x,t) \approx \sum_{n=1}^{M} a_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

of the 1D heat equation ($k = L = 1$),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = 100.$$

Solutions are plotted at intervals of $\Delta t = 0.05$. **Top:** $M = 100$ terms used in Fourier expansion. **Bottom:** $M = 1000$ terms.

of heat conduction,

$$\phi(x,t) = -K_0 \frac{\partial u}{\partial x}(x,t). \tag{90}$$

We may view this situation in the following way: The rod is taken out of an oven, where it was heated to a constant temperature of 100 throughout, and then inserted between two giant "heat sinks", e.g., enormous blocks of ice, that are kept at zero temperature. The initial temperature temperature distribution of the entire system, i.e., rod plus heat bath, is then given by

$$u(x,0) = \begin{cases} 100, & 0 \le x \le L, \\ 0, & \text{otherwise.} \end{cases} \tag{91}$$

As a result, $u(x,0)$ is not differentiable at $x = 0$ or $x = L$ – it may be viewed as having infinite derivatives: $+\infty$ at $x = 0$ and $-\infty$. This is reflected in the solutions $u(x,t)$ for small time: as $t \to 0^+$, the slopes of the function $u(x,t)$ near $x = 0$ and $x = L$ are getting larger and larger in magnitude, implying fluxes that are growing in magnitude.

A more realistic model would have to limit the magnitude of the flux, perhaps to some threshold value ϕ_{max} characteristic of the medium/media. In such a case, however, the resulting heat equation would become a nonlinear PDE in $u(x,t)$, in which case the above method of separation of variables, which relied on both the separability as well as the linearity of the PDE, would not be applicable. As a result, we would probably not be able to derive any closed form solutions to the PDE.

That being said, we could still resort to numerical methods to solve such nonlinear PDEs. If time permits, we'll discuss numerical methods briefly later in the course. Numerical methods for PDEs are discussed in some undergraduate Computational Mathematics (CM) courses.

Finally, having acknowledged the limitations of the linear model in the above example, we expect it to be reasonable in "less extreme" cases, i.e., those cases in which the initial distribution $f(x)$ is not discontinuous, and possibly differentiable, or at least piecewise differentiable, with bounded derivatives (or one-sided derivatives).

Heat equation with zero-flux conditions

`Section 2.4 of text by Haberman`

We now consider the 1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{92}$$

with zero-flux boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}u(L,t) = 0, \tag{93}$$

and initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L. \tag{94}$$

Recall that a zero-flux condition at an endpoint implies that there is no heat flux through the endpoint, i.e., no heat flows out and no heat flows in. This can be caused by an *insulated endpoint*. In this problem, both endpoints are insulated.

Without even solving this problem formally, we can determine the asymptotic behaviour of the solution to this problem. The fact that the endpoints are insulated implies that no heat will escape from the rod. Recall that the equilibrium temperature distribution for an insulated rod is the constant solution $u_{eq}(x) = C$. In fact, we can say more than this – we derived that

$$u_{eq}(x) = \frac{1}{L}\int_0^L f(x)\,dx, \tag{95}$$

i.e., $u_{eq}(x)$ is the average value of the initial temperature distribution. We'll see that our formal solution $u(x,t)$ in terms of Fourier series will yield this result.

Once again, we try a separation of variables solution,

$$u(x,t) = \phi(x)G(t). \tag{96}$$

This implies that the boundary conditions to be satisfied by $\phi(x)$ are

$$\phi'(0) = 0, \qquad \phi'(L) = 0. \tag{97}$$

As before, substitution into the PDE yields

$$\phi G' = k\phi'' G. \tag{98}$$

This can be cast into separated form

$$\frac{G'}{kG} = \frac{\phi''}{\phi} = \mu \tag{99}$$

where $\mu \in \mathbf{R}$ is the separation constant. The equations for G and ϕ become

$$G' - \mu k G = 0, \qquad \phi'' - \mu\phi = 0. \tag{100}$$

Once again, we examine the three cases for μ:

1. **Case 1:** $\mu > 0$ The general solution to the ϕ-equation will be

   $$\phi(x) = C_1 e^{\sqrt{\mu}x} + C_2 e^{-\sqrt{\mu}x}. \tag{101}$$

   From

   $$\phi'(x) = \sqrt{\mu} C_1 e^{-\sqrt{\mu}x} - \sqrt{\mu} C_2 e^{-\sqrt{\mu}x}, \tag{102}$$

   the condition $\phi'(0) = 0$ implies that $C_1 = C_2$, i.e.,

   $$\phi(x) = 2C_1 \cosh(\sqrt{\mu}x). \tag{103}$$

   The condition $\phi'(0) = 0$ implies that

   $$\phi'(x) = 2\sqrt{\mu} C_1 \sinh(\sqrt{\mu}L) = 0, \tag{104}$$

   which is possible only if $C_1 = 0$. As a result, the only solution for this case is the trivial solution $\phi(x) = 0$.

2. **Case 2:** $\mu = 0$ The general solution to the ϕ-equation will be

   $$\phi(x) = C_1 x + C_2. \tag{105}$$

   Thus $\phi'(x) = C_1$. The condition $\phi'(0) = 0$ implies that $C_1 = 0$. Therefore $\phi(x) = C_2$, which satisfies the other boundary condition. Therefore there is a nontrivial solution. Up to a constant, then, the solution is $\phi(x) = 1$.

3. **Case 3:** $\mu < 0$ We let $\mu = \lambda$, where $\lambda > 0$. The general solution to the ϕ-equation will be

   $$\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x). \tag{106}$$

   From

   $$\phi'(x) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda} C_2 \cos(\sqrt{\lambda}x), \tag{107}$$

the condition $\phi'(0) = 0$ implies that $C_2 = 0$, i.e.,

$$\phi(x) = C_1 \cos(\sqrt{\lambda} x). \tag{108}$$

The condition $\phi'(L) = 0$ implies that

$$\sin(\sqrt{\lambda} L) = 0, \tag{109}$$

$$\sqrt{\lambda} L = n\pi, \qquad n = 1, 2, \cdots, \tag{110}$$

or

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, \cdots. \tag{111}$$

Note that we omit the case $n = 0$, since it would imply that $\lambda = 0$, which we have already considered in Case 2. The result is an infinite set of discrete, positive eigenvalues λ_n. The associated eigenfunctions are:

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right). \tag{112}$$

The reader can sketch the first few eigenfunctions (the first three were sketched in class) - in all cases, the slopes of $\phi_n(x)$ at $x = 0$ and $x = L$ are zero, as expected.

Associated with these eigenvalues/eigenfunctions are the time-dependent solutions

$$G_n(t) = e^{-k\lambda_n t} = e^{-k(n\pi/L)^2 t}. \tag{113}$$

Before discussing the general solution of this problem, we return to the one other eigenvalue, i.e., $\lambda_0 = 0$, with associated eigenfunction $\phi_0(x) = 1$. The associated time-dependent solution will be

$$G_0(t) = e^{-k\lambda_0 t} = 1. \tag{114}$$

In other words, there is no time-variation in this solution.

In summary, the method of separation of variables has produced a discretely infinite set of solutions to the heat equation with zero-flux boundary conditions in Eq. (92). They have the form

$$u_n(x,t) = \phi_n(x) G_n(t) = \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}, \qquad n = 0, 1, 2, \cdots, \tag{115}$$

This implies that any linear combination of these functions, for example,

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{M} c_n u_n(x,t) \\ &= \sum_{n=0}^{M} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-k(n\pi/L)^2 t}, \end{aligned} \tag{116}$$

is also a solution to the heat equation satisfying the zero-flux boundary conditions. Note that for any M, all solutions $u_n(x,t)$, with the exception of $u_0(x,t)$, tend to zero as $t \to \infty$. In fact,

$$u(x,t) \to c_0 \quad \text{as} \quad t \to \infty. \tag{117}$$

The reader might suspect that c_0 will be the equilibrium solution $u_{eq}(x)$ associated with this problem, given in Eq. (95). This is indeed the case, as we'll see in the next lecture, where we connect the above solution with the initial condition $f(x)$.