

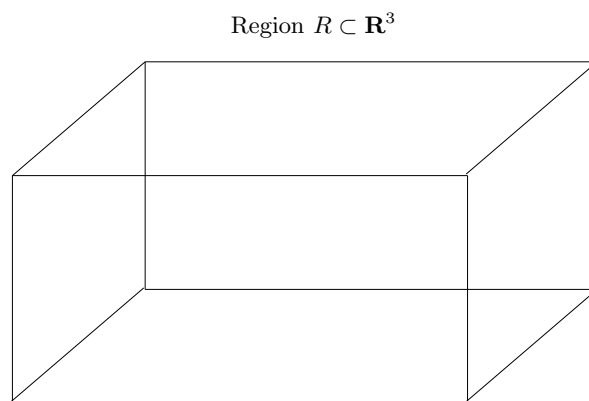
## Gauss' Divergence Theorem (cont'd)

### Conservation laws and some important PDEs yielded by them

Relevant section of AMATH 231 Course Notes: Section 3.1.1

Let  $R$  be a region in  $\mathbf{R}^3$  that will represent our “system” of concern. Our “system” could assume many forms, e.g.,

1. A closed container which contains air or other gases,
2. A tank containing a liquid, e.g., water.



We are concerned with the concentration, or density, (amount per unit volume) of a particular substance,  $X$ , throughout the region. For example,  $X$  could be

1. A particular gas or vapour in the container of gases, e.g., perfume.
2. A particular chemical, e.g., salt, dissolved in the water in the tank.
3. The thermal energy, or heat content, in a material which extends over region  $R$ . This material could be a solid.

We shall let  $u(\mathbf{x}, t) = u(x, y, z, t)$  denote the concentration/density of substance  $X$  at a point  $\mathbf{x} \in R$ .

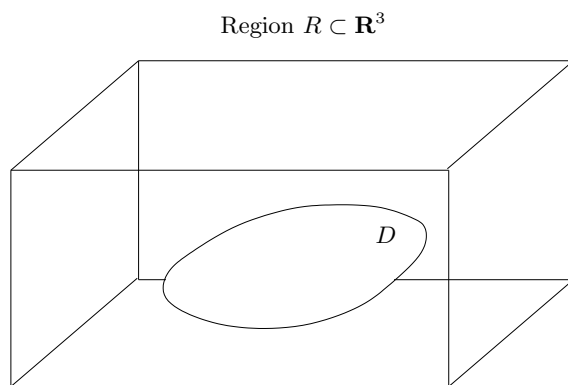
**Note:** The concentration of a physical substance at a point  $\mathbf{x}$  is a mathematical idealization since physical “things” do not exist as points. Such an idealization was employed earlier in this course when we considered point charges and point masses. It would be more accurate to state that  $u(\mathbf{x}, t)$  represents the concentration of substance  $X$  in a small region centered at  $\mathbf{x}$ .

If substance  $X$  is distributed homogeneously throughout region  $R$  at a given time  $a$ , i.e.,  $u(\mathbf{x}, a) = C$ , constant, then we imagine that nothing will change – at least macroscopically, i.e., ignoring changes at the atomic or molecular level – so that  $u(\mathbf{x}, t) = C$  for all  $t > a$ . But if  $u(\mathbf{x}, t)$  is not uniformly distributed, we expect that  $X$  will move from regions of higher concentration to those of lower concentration through the process of **diffusion**. (We won’t get into the details of the molecular motion behind this process.) And this process will continue until  $X$  is distributed uniformly throughout region  $R$ .

The motion of substance  $X$  at any point  $\mathbf{x} \in R$  at a time  $t$  is described by the **flux density vector**,  $\mathbf{j}(\mathbf{x}, t)$ , which characterizes the rate of flow of substance  $X$  per unit area. Being a vector, it will contain the following information:

- the **direction** of transport,
- the **rate of transport** of substance  $X$ .

Now consider an **arbitrary** region  $D \subset R$  enclosed by a surface  $S$  which is assumed to be piecewise  $C^1$  (just like the region and boundary considered in the Divergence Theorem). We shall refer to the surface  $S$  as  $\partial D$ , in order to constantly remind ourselves that it is the boundary of  $D$ . We assume that  $D$ , and hence  $\partial D$ , do not change in time.



Let  $U(t)$  denote the total amount of substance  $X$  in region  $D$  at time  $t$ . From the definition of

$u(\mathbf{x}, t)$ , the concentration (mass per unit volume) of X at  $\mathbf{x}$  at time  $t$ ,

$$U(t) = \int \int \int_D u(\mathbf{x}, t) dV, \quad (1)$$

we are performing an integration of the concentration of X **at a fixed time**  $t$  over the fixed spatial region  $D$ .

The rate of change in time of  $U(t)$  is given by

$$\begin{aligned} U'(t) &= \frac{\partial}{\partial t} \int \int \int_D u(\mathbf{x}, t) dV \\ &= \int \int \int_D \frac{\partial u}{\partial t}(\mathbf{x}, t) dV. \end{aligned} \quad (2)$$

**Aside:** The final line follows from Leibniz' rule for differentiation of integrals since the region  $D$  remains constant in time. Just in case you are not familiar with this rule, we can derive the above result by going back to first principles, i.e.,

$$\begin{aligned} U'(t) &= \lim_{h \rightarrow 0} \left[ \frac{U(t+h) - U(t)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \int \int \int_D u(\mathbf{x}, t+h) dV - \int \int \int_D u(\mathbf{x}, t) dV \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int \int \int_D [u(\mathbf{x}, t+h) - u(\mathbf{x}, t)] dV \\ &= \lim_{h \rightarrow 0} \int \int \int_D \frac{u(\mathbf{x}, t+h) - u(\mathbf{x}, t)}{h} dV \\ &= \int \int \int_D \lim_{h \rightarrow 0} \left[ \frac{u(\mathbf{x}, t+h) - u(\mathbf{x}, t)}{h} \right] dV \\ &= \int \int \int_D \frac{\partial u}{\partial t}(\mathbf{x}, t) dV. \end{aligned} \quad (3)$$

We now assume that no new amounts of  $X$  are added to the region  $R$  at any point  $\mathbf{x} \in R$ , i.e., the total amount of  $X$  in the entire region  $R$  remains constant. But that doesn't mean that the amount of  $X$  in region  $D$  remains constant. For example, it is possible that  $D$  contains a higher amount of  $X$  than other regions, in which case there will be a net outward flow of  $X$  from  $D$ . But we now arrive at an important point:

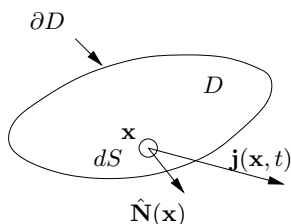
**The only way that  $U(t)$ , the total amount of  $X$  in region  $D$ , can change is by movement of  $X$ , i.e., diffusion, through the boundary  $\partial D$  of  $D$ .**

For this reason, we need only focus on what is happening at points  $\mathbf{x} \in \partial D$ , the boundary of  $D$ .

In what follows, we consider a fixed time  $t$ . Recall that at each point  $\mathbf{x} = (x, y, z) \in \partial D$ , the flow of  $X$  is defined by the flux density vector  $\mathbf{j}(\mathbf{x}, t)$ . The net outward flux of substance  $X$  through an infinitesimal element of surface area  $dS$  centered at point  $\mathbf{x} \in \partial D$  is therefore given by

$$\mathbf{j}(\mathbf{x}, t) \cdot \hat{\mathbf{N}}(\mathbf{x}) dS \quad (4)$$

where  $\hat{\mathbf{N}}(\mathbf{x})$  denotes the unit outward normal vector to the surface  $\partial D$  at  $\mathbf{x}$ . (See sketch below.)



The total outward flux of  $X$  moving through surface  $\partial D$  is therefore given by

$$\int \int_{\partial D} \mathbf{j}(\mathbf{x}, t) \cdot \hat{\mathbf{N}}(\mathbf{x}) dS \quad (5)$$

Note that this is a surface integral. For reasons that will become clear below, it will be advantageous to transform this integral into a volume integral. This is possible by means of the Divergence Theorem. We'll have to assume that the derivatives of the flux vector  $\mathbf{j}$  are continuous over region  $D$ . (More on this later.) With this assumption, we apply the Divergence Theorem to obtain the result,

$$\int \int_{\partial D} \mathbf{j}(\mathbf{x}, t) \cdot \hat{\mathbf{N}}(\mathbf{x}) dS = \int \int \int_D \vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) dV. \quad (6)$$

(Recall that time  $t$  is held fixed.)

We now come to the next major point of this discussion: By conservation of substance X,

The rate of change of the total amount of X in region  $D$  is equal to the negative total outward flux of X through its boundary  $\partial D$ .

Mathematically,

$$U'(t) = - \int \int \int_D \vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) dV . \quad (7)$$

Let's confirm this statement:

- If the net outward flux is positive, then  $U(t)$  is decreasing, i.e.,  $U'(t) < 0$ .
- If the net outward flux is negative, then  $U(t)$  is increasing, i.e.,  $U'(t) > 0$ .
- If the net outward flux is zero, then there is zero instantaneous rate of change in  $U(t)$ , i.e.,  $U'(t) = 0$ .

We now use Eq. (2) in the above equation,

$$U'(t) = \int \int \int_D \frac{\partial u}{\partial t}(\mathbf{x}, t) dV = - \int \int \int_D \vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) dV . \quad (8)$$

Both of the integrals in the above equation involve integration over the fixed region  $D$ , so we can combine them as follows,

$$\boxed{\int \int \int_D \left( \frac{\partial u}{\partial t}(\mathbf{x}, t) + \vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) \right) dV = 0 .} \quad (9)$$

This is known as the **integral form of the conservation law** (for substance X).

We would now very much like to be able to conclude that the integrand is zero at all points  $\mathbf{x}$ , i.e., that conservation applies pointwise. It turns out that we can do so, since we are making the following assumptions:

1. The above relation holds for **any** region  $D \subset R$ , i.e., the region  $D$  is arbitrary.
2. The integrand in the brackets is continuous at all  $\mathbf{x} \in R$ .

## Lecture 21

### Gauss' Divergence Theorem (cont'd)

#### Conservation laws and important PDEs yielded by them (cont'd)

We continue with the discussion from the previous lecture.

**Aside:** The justification for this conclusion, based on the two assumptions given above, is the so-called **du Bois-Reymond Lemma**, which is a multidimensional version of the following result from first-year Calculus.

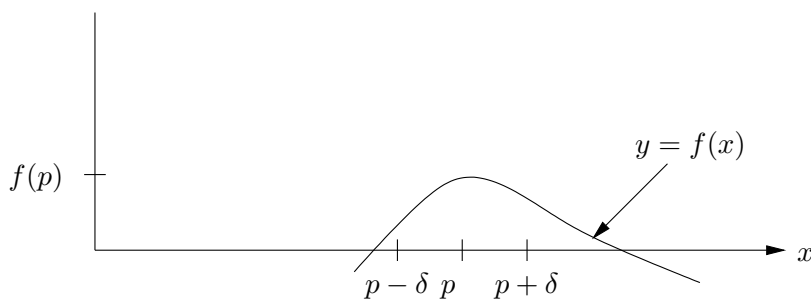
**Du-Bois Reymond Lemma:** Suppose that  $f : \mathbf{R} \rightarrow \mathbf{R}$  is continuous on  $[a, b]$ . Furthermore assume that

$$\int_c^d f(x) dx = 0 \quad (10)$$

for **all** subsets  $[c, d] \subseteq [a, b]$ . Then

$$f(x) = 0 \quad \text{for all } x \in [a, b]. \quad (11)$$

**Sketch of Proof:** By contradiction. Given that Eq. (10) holds, let us assume that there exists a  $p \in (a, b)$  such that  $f(p) \neq 0$ . Without loss of generality, assume that  $f(p) > 0$ . From the continuity of  $f$ , there exists an interval  $(p - \delta, p + \delta)$  over which  $f$  is positive, as sketched below.



It follows that

$$\int_{p-\delta}^{p+\delta} f(x) > 0, \quad (12)$$

which contradicts Eq. (10). Therefore Eq. (11) must hold and the proof is complete.

From the du Bois-Reymond Lemma, we may conclude that the integrand Eq. (9) is zero for all  $x \in R$ , i.e.,

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) + \vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) = 0, \quad (13)$$

which can be written in a more compact form as follows,

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = 0.} \quad (14)$$

This equation is known as the **differential form of the conservation law**. The application of this equation to a number of physical phenomena yields a set of important equations which model them. We consider a few examples below.

1. **Continuity equation:** We let  $u(\mathbf{x}, t) = \rho(\mathbf{x}, t)$  denote the mass density of a fluid (liquid or gas) and  $\mathbf{j} = \rho\mathbf{v}$ , where  $\mathbf{v}$  is the velocity of the fluid. The conservation law (13) becomes

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\mathbf{v}) = 0.} \quad (15)$$

This is known as the **continuity equation** for the fluid. We won't discuss this equation in any detail, except to point out one important special case. When the density of the fluid is constant in space and time, i.e.,  $\rho(\mathbf{x}, t) = \rho_0$ , which is a quite valid assumption for liquids in "normal" situations but not for gases, then Eq. (15) becomes

$$\vec{\nabla} \cdot (\rho_0\mathbf{v}) = \rho_0 \vec{\nabla} \cdot \mathbf{v} = 0 \quad \implies \quad \vec{\nabla} \cdot \mathbf{v} = 0. \quad (16)$$

i.e., the divergence of the velocity vector field is zero. You may recall that a vector field that has zero divergence is often referred to as an **incompressible field**. This is the reason for the terminology.

2. **Diffusion equation:** We now consider the diffusion of a substance X, e.g., a chemical which is dissolved in a solvent. As discussed earlier in this course (in the section dealing with the gradient vector  $\vec{\nabla}f$  associated with a scalar-valued function  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ ), experimental evidence has shown that under "normal" conditions, the process of diffusion is well modelled as follows: If  $u(\mathbf{x}, t)$  denotes the concentration of X at a point  $\mathbf{x}$  at time  $t$ , then the instantaneous movement, or **transport**, of X from  $\mathbf{x}$  is in the direction of **steepest descent** of  $u$  at  $\mathbf{x}$ . Mathematically, this is expressed as follows:

$$\mathbf{j}(\mathbf{x}, t) = -\kappa \vec{\nabla} u(\mathbf{x}, t), \quad (17)$$

where  $\kappa$  is the **diffusivity coefficient** of the solvent in region  $R$ . (Here, we assume that  $\kappa$  is constant throughout  $R$ . It need not be.) Eq. (17) is known as **Fick's Law of Diffusion**. (It is **not** a law, however, but rather a model.)

We now substitute the expression for  $\mathbf{j}$  in Eq. (17) into the second term on the LHS of Eq. (13),

$$\begin{aligned}\vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) &= \vec{\nabla} \cdot (-\kappa \vec{\nabla} u(\mathbf{x}, t)) \\ &= -\kappa \vec{\nabla} \cdot \vec{\nabla} u(\mathbf{x}, t).\end{aligned}\tag{18}$$

Does the RHS make sense? Yes, it does. First of all, the term  $\vec{\nabla} u(\mathbf{x}, t)$  is a **vector** – it is the gradient vector of the concentration function  $u(\mathbf{x}, t)$ . We are then taking the **divergence** of this vector. Let's first investigate this operator for a general scalar valued function  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  by using “del” operator. First of all,

$$\begin{aligned}\vec{\nabla} f &= \left[ \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right] f \\ &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.\end{aligned}\tag{19}$$

We now take the divergence of this vector,

$$\begin{aligned}\vec{\nabla} \cdot \vec{\nabla} f &= \left[ \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right] \left[ \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right] \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned}\tag{20}$$

The result, a scalar, is known as the **Laplacian** of  $f$ . The Laplacian operator is denoted as “ $\nabla^2$ ”, i.e.,

$$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.\tag{21}$$

Substitution of the Laplacian into Eq. (18) yields

$$\vec{\nabla} \cdot \mathbf{j}(\mathbf{x}, t) = -\kappa \nabla^2 u(\mathbf{x}, t).\tag{22}$$

Substitution of this result into Eq. (13) then yields the final result,

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) = \kappa \nabla^2 u(\mathbf{x}, t),\tag{23}$$

which we may write more compactly as

$$\boxed{\frac{\partial u}{\partial t} = \kappa \nabla^2 u}.\tag{24}$$



This equation is known as the (three-dimensional) **Diffusion Equation** for the function  $u(\mathbf{x}, t) = u(x, y, z, t)$ . It is a **partial differential equation** (PDE) since it involves partial derivatives of  $u$  with respect to the position variables,  $x, y, z$  and time  $t$ .

3. **Heat equation:** We now consider the flow of thermal energy, or heat, in a conducting medium. If  $T(\mathbf{x}, t)$  represents the temperature of the medium at a point  $\mathbf{x}$  at time  $t$ , then the relevant scalar field  $u(\mathbf{x}, t)$  is the **heat energy density**, defined as follows,

$$u(\mathbf{x}, t) = c\rho T(\mathbf{x}, t), \quad (25)$$

where  $c$  is the **specific heat** of the medium (the amount of heat required to raise the temperature of a unit mass of medium by one degree) and  $\rho$  is the density of the medium. The vector field  $\mathbf{j}$  is the **heat flux density**. Once again, experimental evidence indicates that under “normal” conditions,

$$\mathbf{j} = -k\vec{\nabla}T, \quad (26)$$

where  $k$  is the thermal conductivity of the medium. This is **Fourier’s law of heat conduction**. (Once again, Fourier’s law indicates that heat flows in the direction of steepest descent of the temperature.)

Substitution of Eqs. (25) and (26) into the conservation equation (14) yields the following partial differential equation for the temperature function  $T(\mathbf{x}, t)$ ,

$$\frac{\partial T}{\partial t} = \kappa\nabla^2T, \quad (27)$$

where  $\kappa = \frac{k}{c\rho}$  is the **thermal diffusivity** of the medium. This PDE is known as the **heat equation**.

## Physical interpretation of the divergence of a three-dimensional vector field $\mathbf{F}$

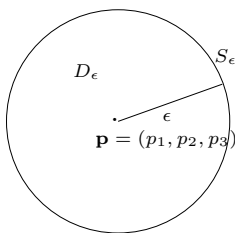
In this section, we shall use the Divergence Theorem to provide a physical interpretation of the divergence of a planar vector field  $\mathbf{F}(x, y, z)$  at a point  $\mathbf{p} = (p_1, p_2, p_3)$ . Our approach will be a quite straightforward extension of the result that we obtained in two dimensions using the Divergence Theorem in the Plane.

First recall the Divergence Theorem: Let  $S$  be a simple closed surface enclosing a region  $D$  in  $\mathbf{R}^3$  and let  $\hat{\mathbf{N}}$  denote the unit outward normal to  $C$ . Also let  $\mathbf{F} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be a vector field which is  $C^1$  on the set  $C \cup D$ , i.e.,  $\mathbf{F}$  has continuous first derivatives at all points of  $C \cup D$ . Then,

$$\int \int_S \mathbf{F} \cdot \hat{\mathbf{N}} dS = \int \int \int_D \left[ \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] dV = \int \int \int_D \vec{\nabla} \cdot \mathbf{F} dV. \quad (28)$$

The leftmost term of the equation is the total outward flux of  $\mathbf{F}$  through the surface  $S$ . The middle and right terms of the equation are the integration of the divergence of  $\mathbf{F}$  over the three-dimensional region  $D$ .

Now let  $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}^3$  be a point of interest and assume that the vector field  $\mathbf{F}(x, y)$  is  $C^1$  in a suitably large three-dimensional neighbourhood of  $\mathbf{p}$ . For an  $\epsilon > 0$ , let  $S_\epsilon$  be the spherical surface of radius  $\epsilon$  centered at  $\mathbf{p}$ . Furthermore, let  $D_\epsilon$  be the region enclosed by  $S_\epsilon$ . The situation is sketched below.



From the Divergence Theorem, the total outward flux of  $\mathbf{F}$  through the surface  $S_\epsilon$  is given by

$$\begin{aligned} \int \int_{S_\epsilon} \mathbf{F} \cdot \hat{\mathbf{N}} ds &= \int \int \int_{D_\epsilon} \vec{\nabla} \cdot \mathbf{F}(\mathbf{x}) dV \\ &= \vec{\nabla} \cdot \mathbf{F}(\mathbf{q}_\epsilon) V(D_\epsilon) \quad \text{for some point } \mathbf{q}_\epsilon \in D_\epsilon, \end{aligned} \quad (29)$$

where

$$V(D_\epsilon) = \frac{4}{3}\pi\epsilon^3 \quad (30)$$

is the volume of the region enclosed by the surface  $S_\epsilon$ . Here, we have once again used the Mean Value Theorem for Integrals, since the integrand  $\vec{\nabla} \cdot \mathbf{F}$  is continuous on  $D_\epsilon$ .

Once again, we divide both sides by the volume  $V(D_\epsilon)$  to obtain

$$\vec{\nabla} \mathbf{F}(\mathbf{q}_\epsilon) = \frac{1}{V(D_\epsilon)} \int \int_{S_\epsilon} \mathbf{F} \cdot \hat{\mathbf{N}} \, dS. \quad (31)$$

Now take the limit  $\epsilon \rightarrow 0$ , which implies that the surface  $S_\epsilon$  is being shrunk toward the point  $\mathbf{p}$ . As such, the points  $\mathbf{q}_\epsilon$  must approach  $\mathbf{p}$  in the limit. We have the result,

$$\vec{\nabla} \cdot \mathbf{F}(\mathbf{p}) = \lim_{\epsilon \rightarrow 0} \frac{1}{V(D_\epsilon)} \int \int_{S_\epsilon} \mathbf{F} \cdot \hat{\mathbf{N}} \, dS. \quad (32)$$

In other words,

The (three-dimensional) divergence of  $\mathbf{F}$  at point  $\mathbf{p}$  is the limiting total outward flux of  $\mathbf{F}$  per unit volume at  $\mathbf{p}$ .