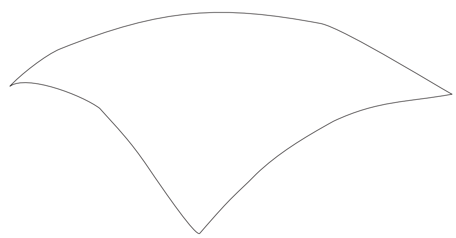
Lecture 17

Surface Integrals

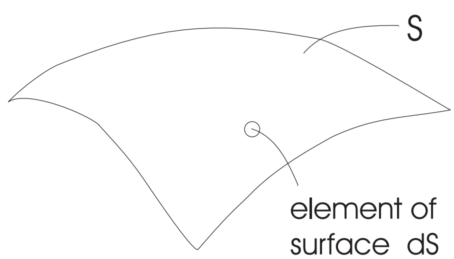
`Relevant section of AMATH 231 Course Notes:  Section 3.1.1`

Surfaces are 2-dimensional subsets of $\mathbf{R^2}$.



We wish to perform two fundamental integration procedures over such surfaces.

1. Integration of scalar functions $f(x,y,z)$ **over** S



$$\text{“}\int_S f\,dS\text{”}$$

Examples:

1. If $f(x,y,z)=1$, then $\int_S f\,dS$ is the area of S.

2. If $f(x,y,z)$ is the charge density (per unit area), then $dq = f\,dS$ is the amount of charge in element dS.

$$\textit{total charge } Q \ = \int_S dq = \int_S f\,dS$$

   If f is the mass density (per unit area) then $dm = f\,dS$, is the amount of charge in element dS,

$$\textit{total mass } M \ = \int_S dm = \int_S f\,dS$$

.

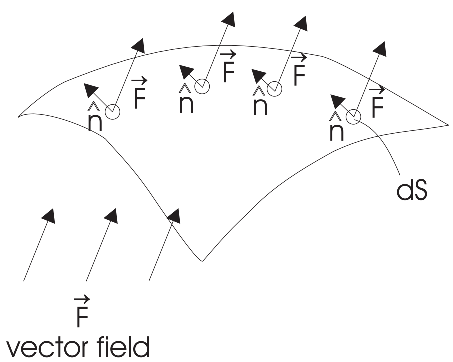
In fact, to illustrate, we can easily compute the surface area of a sphere S_R of radius R from our knowledge of spherical polar coordinates. We'll simply keep $r = R$ fixed to remain on the sphere. The volume integral then becomes a surface integral:

$$\int_{S_R} dS \quad = \quad \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi \; d\phi \; d\theta$$

$$
\begin{aligned}
&= R^2 \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta \\
&= 4\pi R^2. \qquad (1)
\end{aligned}
$$

We'll derive this result more mathematically later in this section.

2. Integration of vector functions $\mathbf{F}(x, y, z)$ over S



At each element dS, compute $\mathbf{F} \cdot \hat{\mathbf{N}}$ where $\hat{\mathbf{N}}$ is the outward unit normal to S.

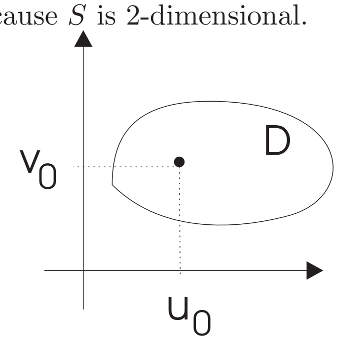
$$\text{“} \int_S \mathbf{F} \cdot \hat{\mathbf{N}} \, dS \text{”}$$

is the total outward flux of $\mathbf{F}$ through S.

In what follows, we shall develop a method to perform integrations over surfaces by using *parametrizations* of surfaces, much as we did for integrations over curves. But we mention that it may be possible to avoid such detailed work if our problem has some nice symmetries. In some special cases that occur in physics such as in gravity and E&M (e.g., spherically symmetric vector fluids and spherically symmetric surfaces), we can compute fluxes in a rather straight forward way using the definition of the surface integral.

Surface integration via parametrization of surfaces

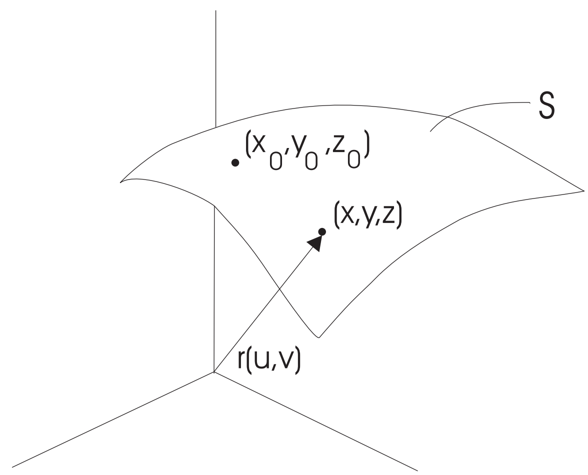
In general, we parametrize the surface S and then express the surface integrals from (1.) and (2.) above as integrations over these parameters. We shall need *two* parameters, say u and v, to define S, because S is 2-dimensional.



D is the set of parameter values (u, v) needed to define S.

The parameterization will be denoted by (to conform with the AMATH 231 Course Notes)

$$\mathbf{g}(u, v) = (x(u, v), y(u, v), z(u, v))$$



$$(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$$

Our goal: We want to write

$$\int_S f\, dS = \int\int_D \underbrace{f(\mathbf{g}(u,v))}_{\substack{f \text{ evaluated} \\ \text{on surface } S}} \underbrace{------}_{\substack{\text{may need} \\ \text{something} \\ \text{here}}} dudv$$

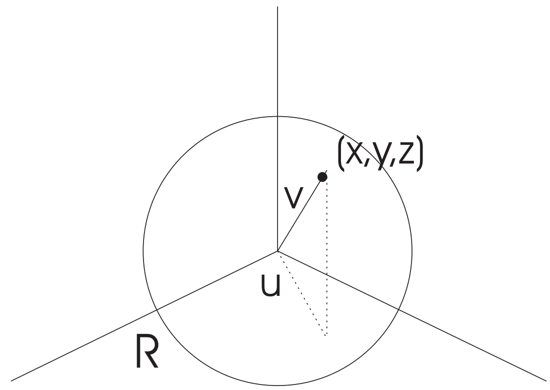
and

$$\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \int\int_D \mathbf{F}(\mathbf{g}(u,v)) \cdot \hat{\mathbf{N}}(u,v) \;-----\; dudv$$

Some examples of surface parameterizations

1. Sphere with radius R, $x^2 + y^2 + z^2 = R^2$.

   Use the angles from spherical polar coordinates to identify a point on the sphere:
   Let $u = \theta$, the azimuthal angle (angle with respect to xz plane).
   Let $v = \phi$, the polar angle (angle with respect to z axis).



   Therefore,

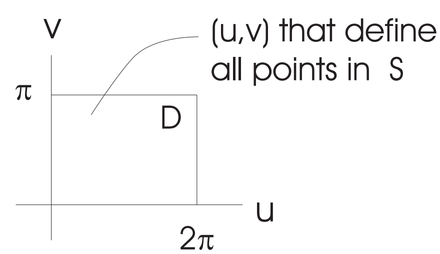
$$
\begin{aligned}
x(u,v) &= R \sin v \cos u \\
y(u,v) &= R \sin v \sin u \\
z(u,v) &= R \cos v,
\end{aligned}
$$

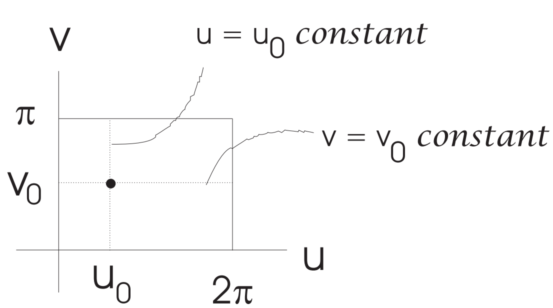
   so that our parametrization may be expressed as

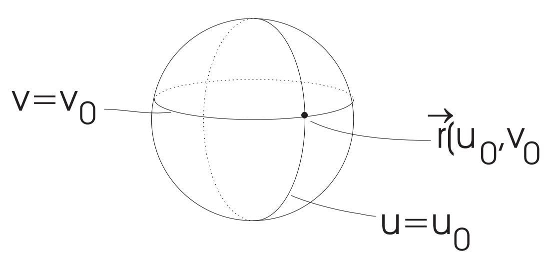
$$\mathbf{g}(u,v) = (R\cos u \sin v, R \sin u \sin v, R\cos v). \tag{2}$$

   The parameter space D that defines the sphere:

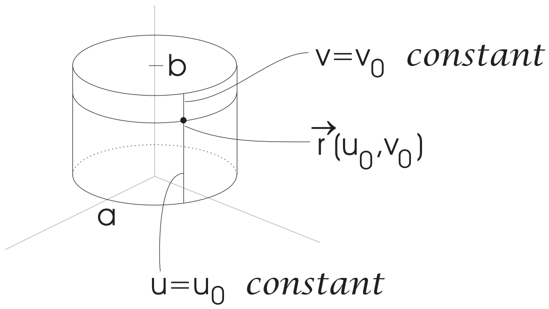
$$0 \leq u \leq 2\pi, 0 \leq v \leq \pi$$







2. Cylinder with $x^2 + y^2 = a^2$, $0 \le z \le b$



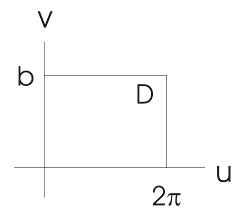
$x(u, v) = a \cos u$

$y(u, v) = a \sin u$

$z(u, v) = v$

$0 \le u \le 2\pi,$

$0 \le v \le b$

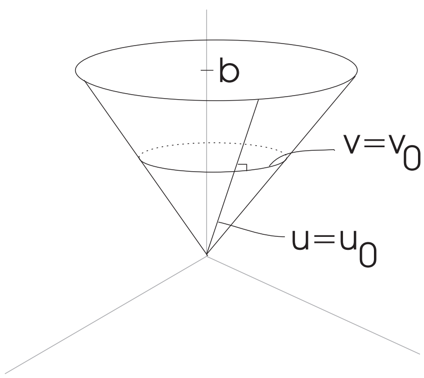


The resulting parametrization:

$$\mathbf{g}(u, v) = (a \cos u, a \sin u, v). \tag{3}$$

The region D in parameter space defining the cylinder is shown at the upper right.

3. Cone with $z^2 = x^2 + y^2$, $0 \le z \le b$



If we set $z = v$ (then $0 \le v \le b$), then

$$x = v \cos u$$

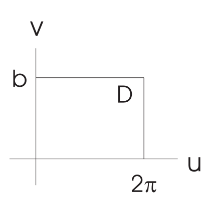
$$y = v \sin u$$

$$0 \le u \le 2\pi, 0 \le v \le b$$

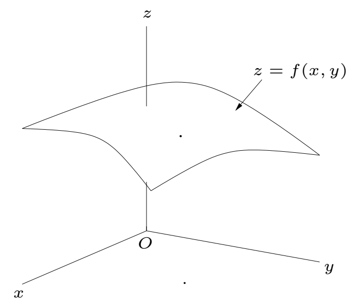
The resulting parametrization:

$$\mathbf{g}(u, v) = (v \cos u, v \sin u, v). \tag{4}$$

The region D in parameter space defining the cone is shown below:

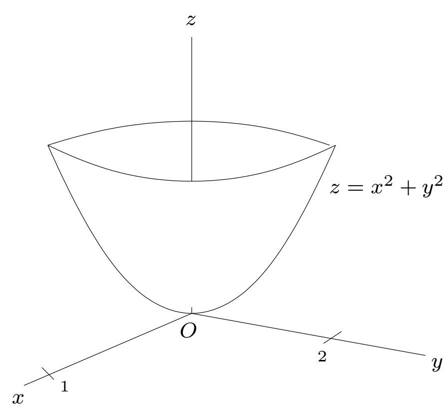


4. The graph of a function $z = f(x, y)$ which, in general, describes a surface in $\mathbf{R^3}$, as sketched generically below.



We could simply use x and y as the parameters – and will do so in some cases in the future. But for now, we'll let u and v be the parameters and let $x = u$ and $v = y$.

For example, consider the graph of $f(x, y) = x^2 + y^2$ over the region $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ in the plane, as sketched below.



The parametrization of this surface is

$$x(u, v) = u\,, \qquad y(u, v) = v \qquad z(u, v) = u^2 + v^2\,, \tag{5}$$

which can be expressed in vector form as

$$\mathbf{g}(u, v) = (u, v, u^2 + v^2)\,. \tag{6}$$

The parameter space D for this example is

$$-1 \leq u \leq 1\,, \qquad -1 \leq v \leq 1\,, \tag{7}$$

the square of sides of length 2 centered at the origin.

5. Planes are the simplest objects because they are flat.

   (a) Any plane parallel to a coordinate axis is trivial to parametrize. Consider, for example, the plane $z = c$. This means that $z(u, v) = c$ for all (u, v). And x and y can simply be parametrized as $x(u, v) = u$ and $y(u, v) = v$. The result:

   $$\mathbf{g}(u, v) = (u, v, c). \tag{8}$$

   The other planes, e.g., $x = a$, are treated in similar ways.

   (b) As for the more general plane, $Ax + By + Cz = D$, we simply express one coordinate in terms of the other two, e.g.,

   $$z = \frac{D}{C} - \frac{A}{C}x - \frac{B}{C}y \qquad \text{(assuming that } \; C \neq 0\text{)}. \tag{9}$$

   Once again, we let x and y be parametrized by u and v, respectively. The resulting parametrization of the general plane is

   $$\mathbf{g}(u, v) = \left(u, v, \frac{D}{C} - \frac{A}{C}u - \frac{B}{C}v \right) . \tag{10}$$

   Please see Example 3.1 and Exercise 3.1 on pages 66-67 of the AMATH 231 Course Notes for further discussion and another method of generating planes.

From AMATH 231 Course Notes

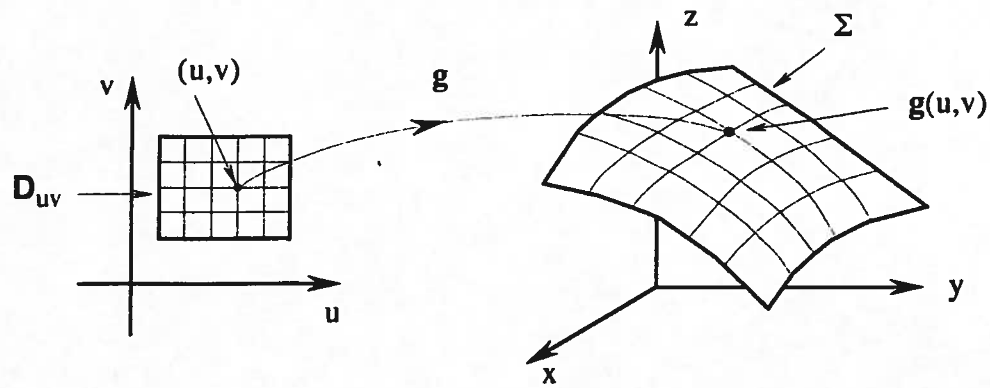


Figure 3.1: The vector-valued function $\mathbf{g}$ maps the domain $\mathcal{D}_{uv}$ onto the surface Σ.

Example 3.1:
The equation

$$\mathbf{x} = \mathbf{g}(u, v) = \mathbf{a} + u\mathbf{e}_1 + v\mathbf{e}_2, \quad (u, v) \in \mathcal{D}_{uv}, \tag{3.2}$$

where $\mathcal{D}_{uv} = \{(u, v) \mid -1 \le u, v \le 1\}$, and $\mathbf{e}_1, \mathbf{e}_2$ are two linearly independent vectors in $\mathbb{R}^3$, describes a surface which is a piece of the plane through the point $\mathbf{a}$, and containing the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Referring to Figure 3.2, the vector $\mathbf{x} - \mathbf{a}$ lies in the plane and hence is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$.

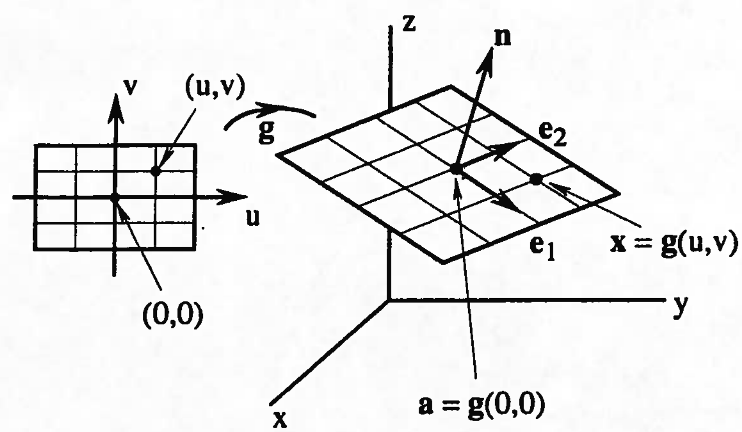


Figure 3.2: Parametric representation of a plane.

One can obtain the equation of the plane in standard form

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0 \tag{3.3}$$

by calculating a normal vector $\mathbf{n}$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the plane the vector product $\mathbf{e}_1 \times \mathbf{e}_2$ is a vector normal to the plane:

$$\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2.$$

Then, taking the scalar product of equation (3.2) with $\mathbf{n}$ gives

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{a}) = 0,$$

which is the standard form (3.3).

Recall: The vector product $\mathbf{a}\times\mathbf{b}$ can be evaluated using a "symbolic determinant":

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{3.4}$$

which, when formally expanded by the first row, gives

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \quad \square$$

Exercise 3.1:

A plane in $\mathbb{R}^3$ is given by

$$\mathbf{x} = (1,2,3) + u(1,1,0) + v(1,-1,1),$$

with $(u,v)\in\mathbb{R}^2$. Find the equation of the plane in standard form.

Answer: $(x-1)-(y-2)-2(z-3)=0.$

Normal vector: N = (1,1,0) X (1,-1,1) = (1,-1,-2)
Vector from (1,2,3) to (x,y,z) on plane is v = (x-1,y-2,z-3)
N . v = 0 => 1(x-1) + (-1)(y-2) + (-2)(z-3) = 0

Exercise 3.2:

Find a vector-valued function to describe the plane $x-3y+z=2$.
Hint: Let $u=x$, $v=y$.

z = 2-x+3y=2-u+3v
so (x,y,z) = (u,v,2-u+3v)

Answer: $\mathbf{x} = \mathbf{g}(u,v) = (0,0,2) + u(1,0,-1) + v(0,1,3). \quad \square$

Example 3.2:

The vector-valued function $\mathbf{g}:\mathcal{D}_{uv}\to\mathbb{R}^3$ defined by

$$\mathbf{g}(u,v) = (\sin u\cos v, \sin u\sin v, \cos u), \tag{3.5}$$

with

$$\mathcal{D}_{uv} = \{(u,v) \mid 0\le u\le\pi, \quad 0\le v\le 2\pi\},$$

describes the surface of the unit sphere in $\mathbb{R}^3$. This can be verified by writing $\mathbf{x}=\mathbf{g}(u,v)$ and verifying that

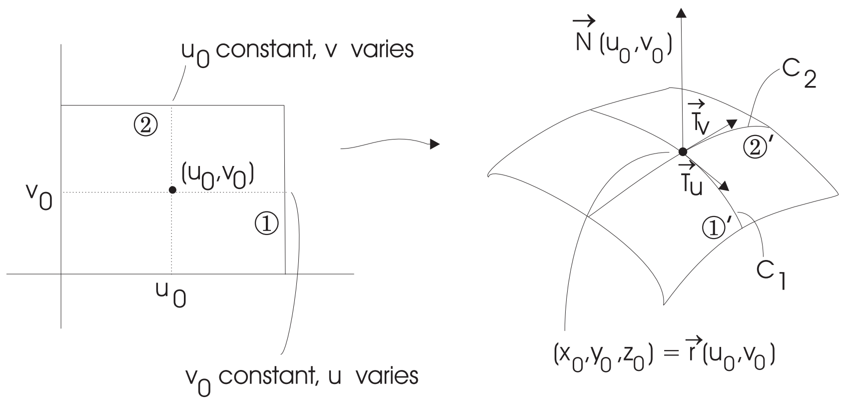
$$\|\mathbf{x}\|^2 = x^2+y^2+z^2 = 1.$$

The vector-valued function (3.5) is obtained from the formulas that relate spherical polar coordinates to Cartesian coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned}$$

Normal vectors to surfaces from parametrizations

`Relevant sections from AMATH 231 Course Notes: Sections 3.1.2 and 3.1.3`



Curve C_1 on the surface is obtained by fixing $v = v_0$ and letting u vary:

$$\mathbf{c}_{v0(u)} = \mathbf{g}(u, v_0).$$

Curve C_2 on the surface is obtained by fixing $u = u_0$ and letting v vary:

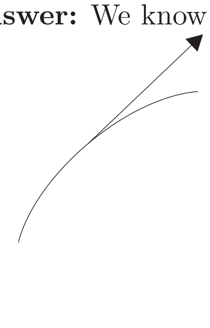
$$\mathbf{c}_{u_0}(v) = \mathbf{g}(u_0, v).$$

Goal: To find $\mathbf{N}(u_0, v_0)$, the normal vector to S at $(x_0, y_0, z_0) = \mathbf{g}(u_0, v_0)$. $\mathbf{N}$ is perpendicular to the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ to curves $\mathbf{c}_{v_0}(u)$ and $\mathbf{c}_{u_0}(v)$, respectively.

We can choose either of $\mathbf{N} = \pm \mathbf{T}_u \times \mathbf{T}_v$.

Question: How do we find $\mathbf{T}_u$ and $\mathbf{T}_v$?

Answer: We know how to obtain tangent vectors to curves:



$$\begin{aligned} \mathbf{T}_u &= \mathbf{c}_{v_0}{}'(u) \\ &= \frac{\partial}{\partial u}\mathbf{g}(u, v)_{v=v_0} \end{aligned}$$

$$\Rightarrow \mathbf{T}_u(u_0, v_0) = \left.\frac{\partial}{\partial u}\mathbf{g}(u, v)\right|_{(u, v) = (u_0, v_0)}$$

$$= \left(\left.\frac{\partial x}{\partial u}\right|_{(u_0, v_0)}, \left.\frac{\partial y}{\partial u}\right|_{(u_0, v_0)}, \left.\frac{\partial z}{\partial u}\right|_{(u_0, v_0)} \right)$$

Likewise,

$$\mathbf{T}_v(u_0, v_0) = \left(\left.\frac{\partial x}{\partial v}\right|_{(u_0,v_0)}, \left.\frac{\partial y}{\partial v}\right|_{(u_0,v_0)}, \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)} \right)$$

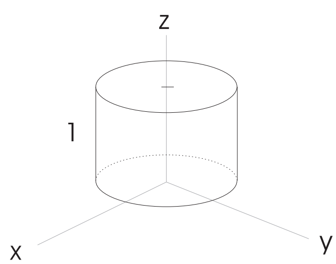
(You require two vectors – one involves $\frac{\partial}{\partial u}$, the other $\frac{\partial}{\partial v}$ – because you have two parameters u, v.)

$$\begin{aligned}
\mathbf{N}(u_0, v_0) &= \pm\, \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0) \\
&= \pm \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{matrix} \right|_{(u_0,v_0)}
\end{aligned}$$

Note that all derivatives are evaluated at (u_0, v_0).

Examples:

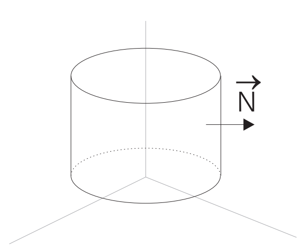
1. The cylinder $\mathbf{g}(u, v) = (\cos u, \sin u, v)$, $0 \le u \le 2\pi$, $0 \le v \le 1$.



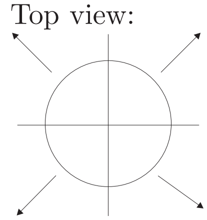
Step 1: Parameterization: Given.

Step 2: Compute $\mathbf{N}(u, v)$ via $\mathbf{T}_u$, $\mathbf{T}_v$.

$$\begin{aligned}
\mathbf{T}_u &= \frac{\partial}{\partial u}\mathbf{g}(u, v) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (-\sin u, \cos u, 0) \\
\mathbf{T}_v &= \frac{\partial}{\partial v}\mathbf{g}(u, v) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0, 0, 1) \\
\Rightarrow \quad \mathbf{T}_u \times \mathbf{T}_v &= \left| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{matrix} \right| = (\cos u, \sin u, 0) = \mathbf{N}(u, v)
\end{aligned}$$



Top view:



As expected, the normal vector $\mathbf{N}$ is parallel to the xy-plane (since it has zero z-component) and points radially outward from the cylindrical surface. Note also that $\|\mathbf{N}(u,v)\| = 1$.

2. The general plane $Ax + By + Cz = D$ which was discussed in the previous section. (Of course, we know the answer: A normal vector to the plane is given by $\mathbf{N} = (A, B, C)$. But let's see what our parametrization method yields.)

   By expressing z in terms of x and y, we arrived at the following parametrization,

$$\mathbf{g}(u,v) = \left(u, v, \frac{D}{C} - \frac{A}{C}u - \frac{B}{C}v\right). \tag{11}$$

   We first compute the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$:

$$\mathbf{T}_u = \frac{\partial}{\partial u}\mathbf{g}(u,v) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \left(1, 0, -\frac{A}{C}\right) \tag{12}$$

$$\mathbf{T}_v = \frac{\partial}{\partial v}\mathbf{g}(u,v) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(0, 1, -\frac{B}{C}\right) \tag{13}$$

$$\Rightarrow \quad \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{A}{C} \\ 0 & 1 & = \frac{B}{C} \end{vmatrix} = \left(\frac{A}{C}, \frac{B}{C}, 1\right) = \mathbf{N}(u,v). \tag{14}$$

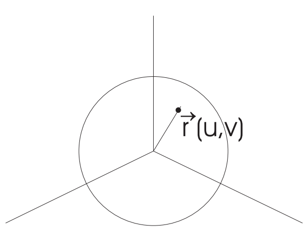
   This result is consistent with the well-known result that the vector (A, B, C) (or any constant multiple of it) is normal to the plane $Ax + By + Cz = D$.

3. Sphere with radius R, $x^2 + y^2 + z^2 = R^2$. Parameterize as $(u = \theta, v = \phi)$ (spherical coordinates).

$$\mathbf{g}(u,v) = (R\underbrace{\cos u \sin v}_{x}, R\underbrace{\sin u \sin v}_{y}, R\underbrace{\cos v}_{z})$$

$$0 \le u \le 2\pi, \qquad 0 \le v \le \pi$$

   Need to compute $\mathbf{T}_u$ and $\mathbf{T}_v$:

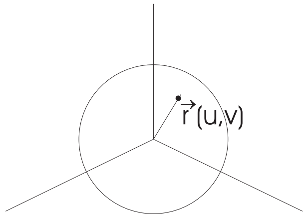


$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin u \sin v & R\cos u \sin v & 0 \\ R\cos u \cos v & R\sin u \cos v & -R\sin v \end{vmatrix}$$

$$= (-R^2 \cos u \sin^2 v, -R^2 \sin u \sin^2 v, \underbrace{-R^2 \sin^2 u \sin v \cos v - R^2 \cos^2 u \sin v \cos v}_{\textit{simplifies to } -R^2 \sin v \cos v})$$

$$= -R \sin v \, \underbrace{(R\cos u \sin v, R \sin u \sin v, R\cos v)}_{\mathbf{g}(u,v)}$$

$$= -R\sin v \; \mathbf{g}(u,v)$$

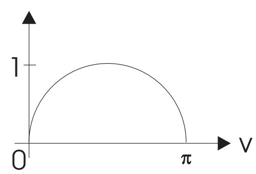


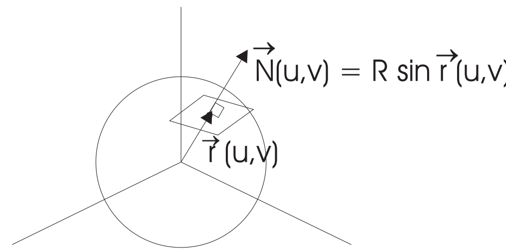
For the outward normal, choose

$$\begin{aligned} \mathbf{N} &= -\mathbf{T}_u \times \mathbf{T}_v \quad (or\ \mathbf{T}_v \times \mathbf{T}_u) \\ &= R \sin v \, \mathbf{g}(u,v) \end{aligned}$$

Note that $\sin v \geq 0$ for $v \in [0, \pi]$ as shown below on the left. The resulting outward normal $\mathbf{N}$ is sketched on the right.

$0 \leq v \leq \pi$





Let us now compute the magnitude of this normal vector:

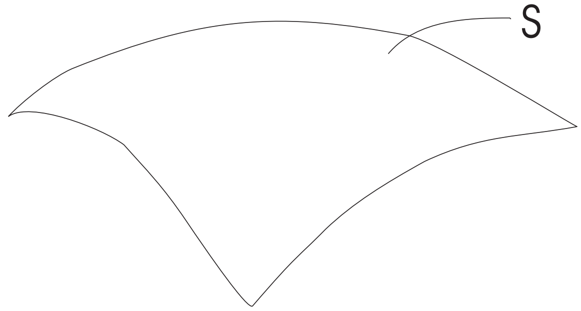
$$\parallel \mathbf{N} \parallel = R|\sin v| \parallel \mathbf{g}(u,v) \parallel = R^2 \sin v, \tag{15}$$

where we have used the fact that $\parallel \mathbf{g}(u,v) \parallel = R$ since the point $\mathbf{g}(u,v)$, by construction, lies on a surface of sphere R. You may now be wondering why the magnitude of this normal vector is not constant; in particular, why would it be changing with the spherical polar angle v (angle between radial vector $\mathbf{g}$ and the z-axis). The answer is that the $R^2 \sin v$ factor provides another piece of information that necessary to translate the integration over the sphere to an integration in uv parameter space. It is the "conversion factor," or **Jacobian** between an infinitesimal element of area dA in the uv parameter space (shown at the left below) and the actual element of area dS on the sphere (shown at the right). We'll show this below.

Lecture 18

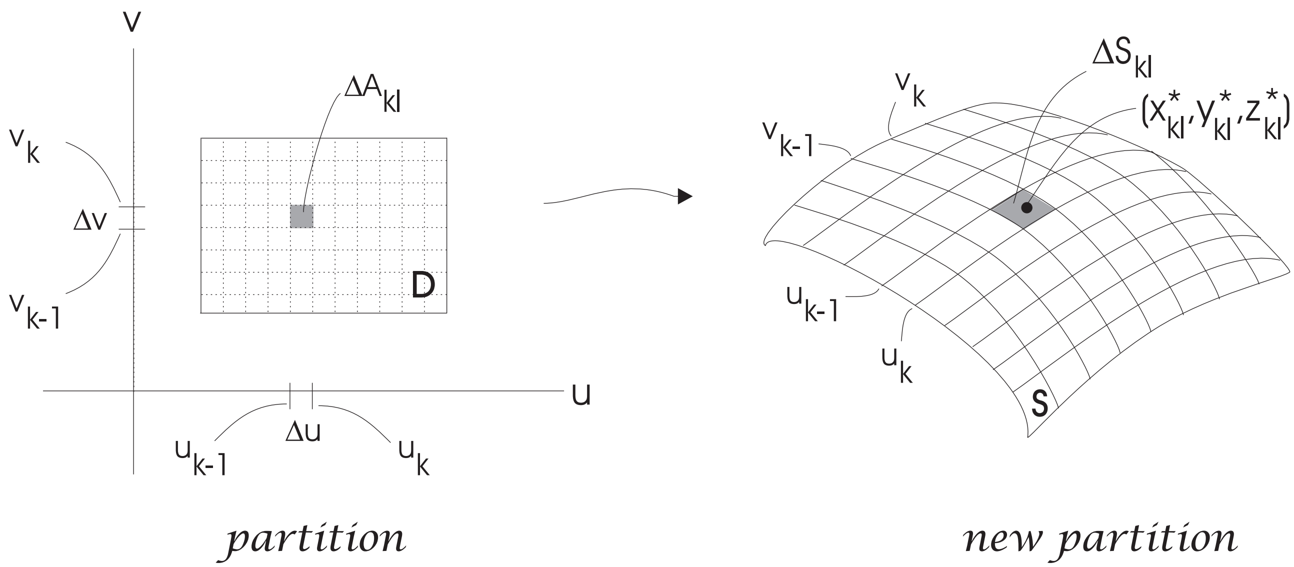
Surface integrals of scalar functions

`Relevant section from AMATH 231 Course Notes: 3.1.2, 3.1.3, 3.2.1`



Let $f(x, y, z)$ be defined over a surface S. We want to integrate f over S.

In the "spirit of calculus":



The partitioning of D on the left defines a partitioning of S shown on the right.

Pick sample points (u^*_k, v^*_k), a subset of an appropriate square. Each of these points defines a point $(x^*_{kl}, y^*_{kl}, z^*_{kl}) = \mathbf{g}(u^*_{kl}, v^*_{kl})$.

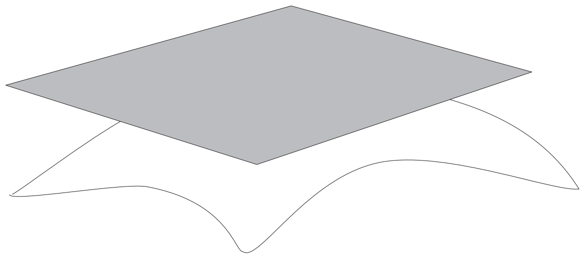
Evaluate f at $(x^*_{kl}, y^*_{kl}, z^*_{kl})$.

Goal:

$$\sum_{kl} f(x^*_{kl}, y^*_{kl}, z^*_{kl}) \Delta S_{kl}$$

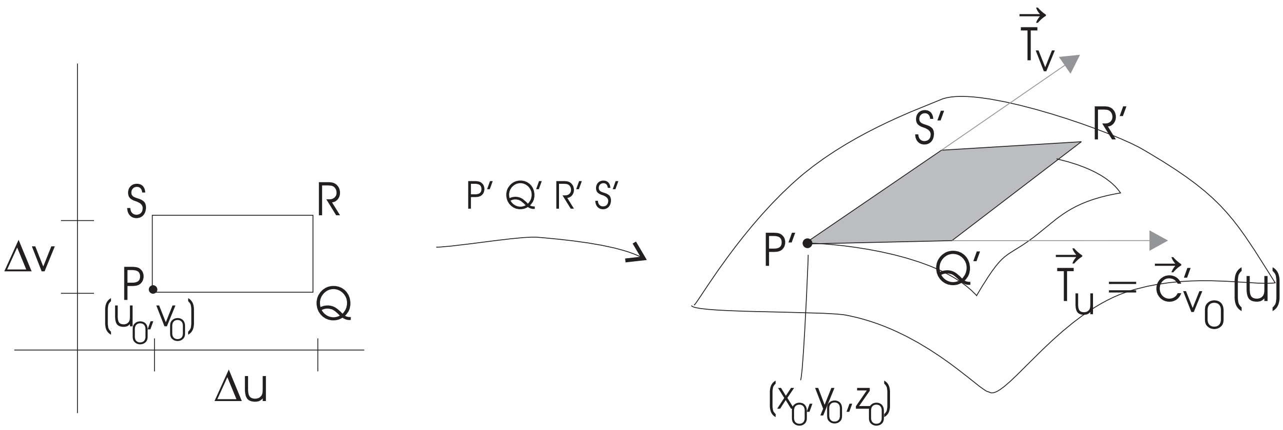
But we don't know what ΔS_{kl} is!

Approximate ΔS_{kl} by $\Delta S_{kl}{}'$ – the area of a flat plane that "covers" ΔS_{kl} that is tangent to S somewhere in ΔS_{kl}. So



$$\cong \sum f(x^*_{kl}, y^*_{kl}, z^*_{kl})\Delta S_{kl}{}'$$

So what is $\Delta S_{kl}{}'$? The area of the *flat* tangent plane piece.



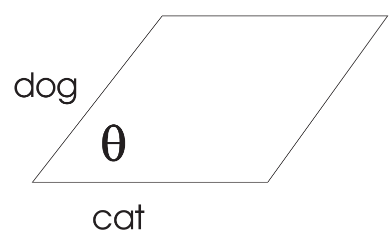
$$\|\overline{P'Q'}\| = (\text{ speed w.r.t. u }) \times \Delta u = \|\mathbf{T}_u(u_0, v_0)\|\Delta u \qquad (cat)$$

Similarly,

$$\|\overline{P'S'}\| = \|\mathbf{T}_v(u_0, v_0)\|\Delta v \qquad (dog)$$

So the area of the parallelogram $\|P'Q'R'S'\|$ is

$$\begin{aligned} &= (cat)\cdot(dog)\cdot\sin\theta \\ &= \|\mathbf{T}_u\|\|\mathbf{T}_v\|\sin\theta\Delta u\Delta v \\ &= \|\mathbf{T}_u\times\mathbf{T}_v\|\Delta u\Delta v \\ &= \|\mathbf{N}(u_0, v_0)\|\Delta u\Delta v \end{aligned}$$



The final result is

$$\sum f(\ldots)\Delta S' \underbrace{\cong}_{\substack{n,m\to\infty \\ partition}} \sum f(x^*, y^*, z^*)\|\mathbf{N}(u^*, v^*)\|\Delta u\Delta v$$

$$\Rightarrow \int_S f\,dS = \int\int_{u,v} \overbrace{f(\mathbf{g}(u,v))}^{\substack{f\ evaluated \\ on\ surface}} \underbrace{\|\mathbf{N}(u,v)\|\underbrace{du\,dv}_{dv\,du}}_{dS\ element\ of\ area}$$

This is the formula for computing surface integrals of scalar-valued functions. Let us now work out a few surface integrations using the result stated at the end of the previous section.

Example 1: Surface area of a sphere with radius R.

We shall use the parametrization introduced previously:

$$\begin{aligned}\mathbf{g}(u,v) &= (x(u,v), y(u,v), z(u,v)) \\ &= (R\cos u \sin v, R\sin u \sin v, R\cos v)\end{aligned}$$

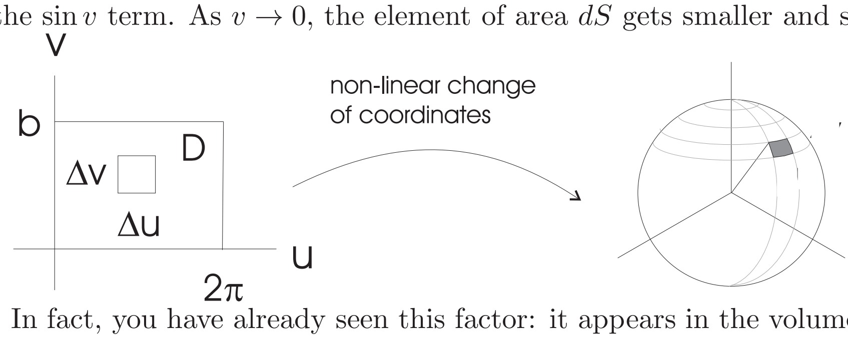
where $v = \theta$, $u = \phi$, $0 \le v \le \pi$, $0 \le u \le 2\pi$. Recall that

$$\mathbf{N} = \pm \mathbf{T}_u \times \mathbf{T}_v$$

We choose $\mathbf{T}_v \times \mathbf{T}_u$ since it is the outward normal. Previously, we computed the magnitude of this normal vector to be

$$\|\mathbf{N}(u,v)\| = R^2 \sin v. \tag{16}$$

Let us now investigate the signifance of this result. In the figure below, on the left, the infintesimal element of area dA produced by infinitesimals du and dv does not vary as we move around the rectangle in uv space. But the resulting element of area dS does vary on the sphere. For example, the closer you are to the top of the sphere (i.e., the "North Pole"), the smaller the element ds because the lines of constant v, i.e., the lines of longitude, are coming together as we approach the pole. This is reflected in the $\sin v$ term. As $v \to 0$, the element of area dS gets smaller and smaller.



$$dA = du dv$$

$$\to dS = \underbrace{\|\mathbf{N}(u,v)\|}_{Jacobian} du dv$$

In fact, you have already seen this factor: it appears in the volume element dV in spherical polar coordinates. The only difference is that the R above, which is a fixed value – since we are considering a spherical shell – becomes the lower case r in 3D spherical polar coordinates, since we also integrate over this variable. In order to compute the surface area of the sphere, we choose $f(x, y, z) = 1$. Then

$$\int\int_S f\, dS = \int\int_S dS = A$$

In general, the integration of a function f over a surface will be computed as follows,

$$\int\int_S f\, dS \to \int\int_D f(\mathbf{g}(u,v))\, \|\mathbf{N}(u,v)\| \underbrace{\underbrace{dA}_{dudv}}_{dvdu}$$

This implies that the surface area A is given by $(f = 1)$,

$$\int\int_D \|\mathbf{N}(u,v)\| dA\,.$$

For our problem,

$$\|\mathbf{N}(u,v)\| = R^2 \sin v\,.$$

We have to integrate this function over the region, $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$, i.e.,

$$A = \int\int_D \|\mathbf{N}(u,v)\| dA = \underbrace{\int_0^{2\pi}\int_0^{\pi}}_{\substack{parameters\\ that\ generate\\ S}} \underbrace{R^2 \sin v}_{\|\mathbf{N}(u,v)\|}\, \underbrace{dvdu}_{dA}$$

Inner Integral:

$$R^2 \int_0^{\pi} \sin v\, dv = R^2(-\cos v)\Big|_0^{\pi} = 2R^2$$

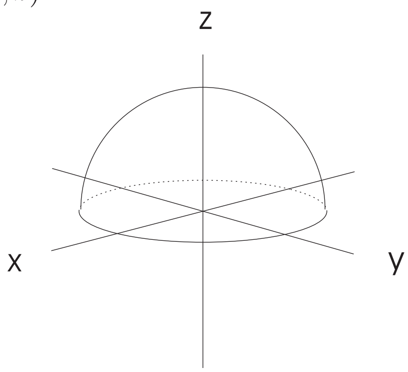
Outer Integral:

$$2R^2 \int_0^{2\pi} du = R^2(u)\Big|_0^{2\pi} = 4\pi R^2$$

.

This agrees with our result for the area of a spherical surface of radius R.

Example 2: Given a thin hemispherical shell, radius R, positioned as shown below. Find its centroid $(0, 0, \overline{z})$.



$$\overline{z} = \frac{\int\int_S z\,dS}{\int\int_S dS}$$

Here, the denominator $\int\int_S dS$ is the surface area of the hemisphere which is $2\pi R^2$.

We need to compute $\int\int_S z\,dS$.

$$\int\int_S z\,dS \underbrace{=}_{\substack{\textit{parametrization}\\ \mathbf{g}(u,v)}} \int\int_D z(u,v)\|\mathbf{N}(u,v)\|dA$$

For $z(u, v)$, we now have to express z on the surface in terms of u and v.

$$\Rightarrow z(u, v) = R\cos v$$

Note that we are integrating over the *upper hemisphere*, so the ranges of the parameters will be $0 \leq u \leq 2\pi$ (as for the sphere) but $0 \leq v \leq \pi/2$ (unlike the sphere).

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \underbrace{R\cos v}_{z} \underbrace{R^2\sin v}_{\substack{\|\mathbf{N}(u,v)\|\\ \textit{Jacobian}}} \quad dvdu$$

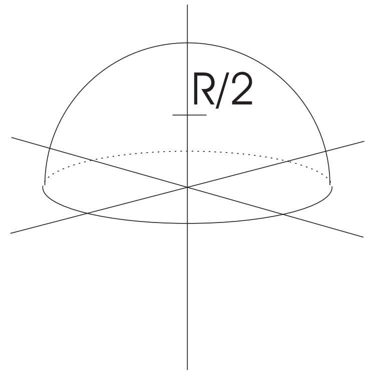
Inner Integral:

$$R^3\int_0^{2\pi}\cos v\sin v\,dv = R^3(\frac{1}{2}\sin^2 v)\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}R^3$$

Outer Integral:

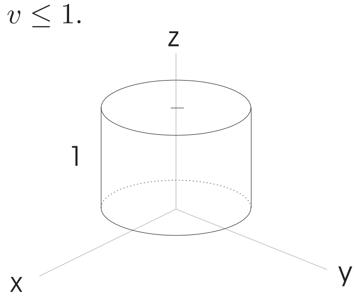
$$\frac{1}{2}R^3\int_0^{2\pi} du = \frac{1}{2}R^3(u)\Bigg|_0^{2\pi} = \pi R^3$$

$$\Rightarrow \overline{z} = \frac{\pi R^3}{2\pi R^2} = \frac{R}{2} \qquad \text{(half-way up)}$$



Note: You may wish to compare this result to the centroid of the solid hemisphere, $\overline{z} = \frac{3}{8}R$, which can be found by integration in 3D spherical polar coordinates.

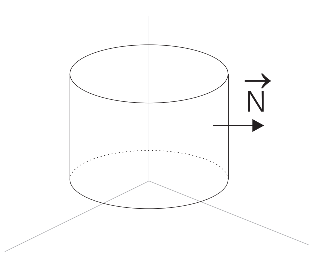
Example 3: Compute the integral $\int\int_S x^2 z\, dS$ over the cylinder $\mathbf{g}(u, v) = (\cos u, \sin u, v)$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$.



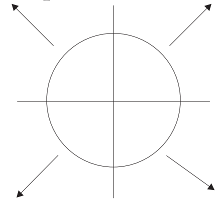
Step 1: Parameterization: Given.

Step 2: Compute $\mathbf{N}(u, v)$ via $\mathbf{T}_u$, $\mathbf{T}_v$.

$$\begin{aligned}
\mathbf{T}_u &= \frac{\partial r}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (-\sin u, \cos u, 0) \\
\mathbf{T}_v &= \frac{\partial r}{\partial v} = (0, 0, 1) \\
\Rightarrow \mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u, \sin u, 0) = \mathbf{N}(u, v)
\end{aligned}$$



Top view:



As expected, the normal vector $\mathbf{N}$ is parallel to the xy-plane (since it has zero z-component) and points radially outward from the cylindrical surface. Note also that $\|\mathbf{N}(u, v)\| = 1$.

We now use the parametrizations $x = \cos u$ and $z = v$ in the integrand to give

$$\int\int_S x^2 z \, dS = \int_0^1 \int_0^{2\pi} \cos^2 u \, v \; dudv = \frac{\pi}{2}. \tag{17}$$

The above integral is particularly convenient to compute since it is *separable*, i.e., the u- and v-dependent terms can be separated so that that the double integral over u and v can be expressed as a product of integrals over u and v:

$$\int_0^1 \int_0^{2\pi} \cos^2 u \, v \, dudv = \left(\int_0^1 v \, dv\right) \left(\int_0^{2\pi} \cos^2 u \, du\right) = \left(\frac{1}{2}\right) (\pi) = \frac{\pi}{2}. \tag{18}$$

Example 4: Compute the area of the cylinder used in Example 3.

We simply have to compute the surface integral $\int\int_S dS$. Using the parametrization from Example 3 and the fact that $\|\mathbf{N}(u,v)\| = 1$, the surface area is given by

$$\int\int_S dS = \int_0^1 \int_0^{2\pi} dudv = 2\pi. \tag{19}$$

This result is confirmed if we take the cylinder, cut it open by means of a vertical cut, and flatten it out. The result is a rectangle with dimensions 2π and 1. The area of this rectangle is 2π.

Example 5: Compute the surface area of a right circular cone (without the base circle) of height h and base radius b. This problem is done in Example 3.4 of the AMATH 231 Course Notes and was covered in class. The appropriate pages of the Course Notes are reproduced below.

In order to approximate the area ΔS of a surface element we need to approximate the vectors $\mathbf{A}$ and $\mathbf{B}$ that define the sides of the surface element in Figure 3.6. Using the linear approximation (3.11) we obtain

$$\mathbf{A} \approx (\Delta u)\frac{\partial \mathbf{g}}{\partial u}(u_0, v_0), \quad \mathbf{B} \approx (\Delta v)\frac{\partial \mathbf{g}}{\partial v}(u_0, v_0).$$

Thus, simplifying $\Delta S \approx \| \mathbf{A} \times \mathbf{B} \|$ gives

$$\Delta S \approx \left\| \frac{\partial \mathbf{g}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{g}}{\partial v}(u_0, v_0) \right\| \Delta u \Delta v. \tag{3.16}$$

To obtain the total area S we have to sum over all surface elements determined by the partition of $\mathcal{D}_{uv}$ and take the limit as $N \to \infty$ and $\max(\Delta u)$, $\max(\Delta v) \to 0$. This process leads to a double integral over the set $\mathcal{D}_{uv}$ in the uv-plane. With the above as motivation we make the following.

Definition:

The *surface area* of the surface by $\mathbf{x} = \mathbf{g}(u, v)$, $(u, v) \in \mathcal{D}_{uv}$, where $\mathbf{g}$ is C^1, is defined by

$$S = \iint\limits_{\mathcal{D}_{uv}} \left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| du\, dv. \tag{3.17}$$

Example 3.4:

Calculate the surface area of a cone of radius b and height h.

Solution: The cone is given by

$$\frac{x^2 + y^2}{b^2} = \frac{z^2}{h^2}, \qquad 0 \le z \le h.$$

A suitable vector-valued function is

$$\mathbf{x} = \mathbf{g}(u, v) = (bu\cos v, bu\sin v, hu), \quad (u, v) \in \mathcal{D}_{uv},$$

with $\mathcal{D}_{uv} = [0, 1] \times [0, 2\pi]$. The tangent vectors are

$$\frac{\partial \mathbf{g}}{\partial u} = (b\cos v, b\sin v, h), \quad \frac{\partial \mathbf{g}}{\partial v} = (-bu\sin v, bu\cos v, 0),$$

giving

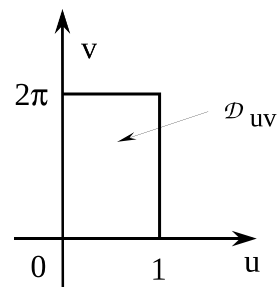
$$\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} = \begin{vmatrix} i & \mathbf{j} & \mathbf{k} \\ b\cos v & b\sin v & h \\ -bu\sin v & bu\cos v & 0 \end{vmatrix} = (-bhu\cos v, -bhu\sin v, b^2 u),$$

and

$$\left\| \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v} \right\| = \sqrt{b^2 + h^2}\, bu.$$

Thus by (3.17) the surface area is

$$\begin{aligned} S &= \iint\limits_{\mathcal{D}_{uv}} \sqrt{b^2+h^2}\, bu\, du\, dv \\ &= \sqrt{b^2+h^2}\, b \int\limits_{u=0}^{1} \left(\int\limits_{v=0}^{2\pi} u\, dv \right) du \\ &= \cdots = \pi b \sqrt{b^2+h^2}. \end{aligned}$$



Comment:

This example is simply intended to show you how the "machinery" works. The surface area of a cone can be calculated by simple geometry. □

The most important part of this subsection is the approximation (3.16) for the area of a surface element, which we shall use when introducing *surface integrals*, the main goal of this chapter.

3.1.4 Orientation of a surface

Consider a surface Σ given by

$$\mathbf{x} = \mathbf{g}(u, v), \qquad (u, v) \in \mathcal{D}_{uv},$$

where $\mathbf{g}$ is one-to-one and of class C^1, and the normal vector

$$\mathbf{N} = \frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}$$

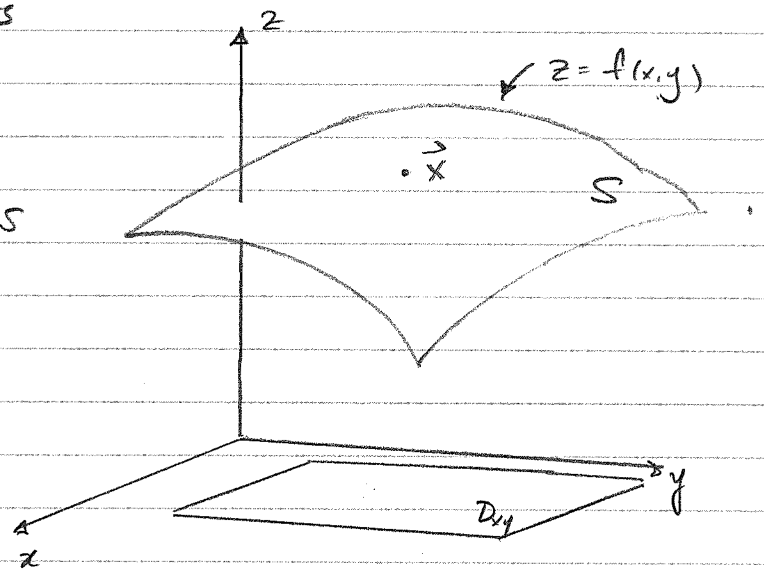
is non-zero for all $(u, v) \in \mathcal{D}_{uv}$. Since $\mathbf{g}$ is one-to-one the surface does not intersect itself (see Figure 3.7), and since $\mathbf{g}$ is of class C^1, $\mathbf{N}$ varies continuously over Σ. In the sequel we shall need to work with a unit normal $\mathbf{n}$ on Σ. There are two choices for $\mathbf{n}$ at each point, namely

$$\mathbf{n} = \pm \frac{1}{\| \mathbf{N} \|} \mathbf{N}.$$

LECTURE 19 INTEGRATION OVER SURFACES (cont'd)

Parametrization of surface defined by $z=f(x,y)$ in terms of x & y

If the surface S over which we wish to integrate is the graph of a function, say $z=f(x,y)$ for $(x,y)\in D$ then it is probably most convenient to use x and y as the parameters. We could set $u=x$ and $v=y$, but let's use x and y in order to emphasize the use of these independent variables as parameters



A point on this surface S will be given by

$$\vec{x} = \vec{g}(x,y) = (x, y, f(x,y))$$

The tangent vectors associated with this parametrization are

$$\vec{T}_u = \vec{T}_x = \frac{\partial \vec{g}}{\partial x} = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$\vec{T}_v = \vec{T}_y = \frac{\partial \vec{g}}{\partial y} = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

Normal vector

$$\vec{N} = \vec{T}_x \times \vec{T}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

The surface area Jacobian will be

$$\|\vec{N}(x,y)\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

Then the area of surface $z = f(x,y)$ for $(x,y) \in D$ is

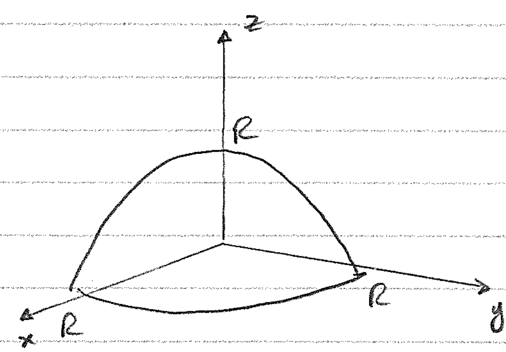
$$A = \iint_S dS = \iint_D \|\vec{N}(x,y)\|\, dx\, dy$$

<u>Example</u>: We consider the upper hemisphere of the sphere

$$x^2 + y^2 + z^2 = R^2$$

ie the function

$$z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$$



(We can't consider the entire sphere -

we wouldn't have a (single-valued) function)

Then $$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1}{\sqrt{R^2 - x^2 - y^2}}(-2x) = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\,\frac{1}{\sqrt{R^2 - x^2 - y^2}}(-2y) = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

Therefore

$$\|\vec{N}(x,y)\| = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right]^{1/2}$$

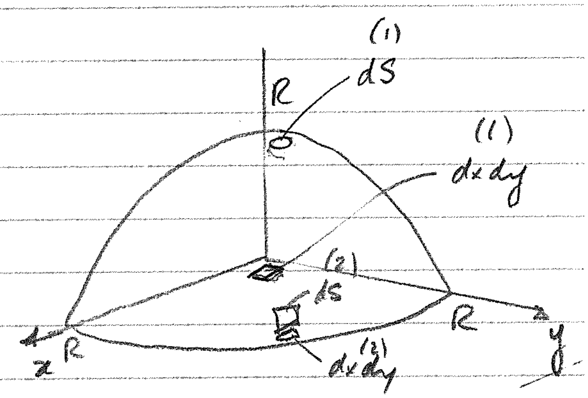
$$= \left[\frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2} + 1 \right]^{1/2}$$

$$= \left[\frac{x^2}{R^2-x^2-y^2} + \frac{y^2}{R^2-x^2-y^2} + \frac{R^2-x^2-y^2}{R^2-x^2-y^2} \right]^{1/2}$$

$$\|\vec{N}(x,y)\| = \frac{R}{\sqrt{R^2-x^2-y^2}}$$

Let's examine this result - it is telling us that an element of surface area dS at the point $(x, y, f(x,y))$ on the surface is related to the area element $dA = dx\,dy$ on the xy-plane as follows:

$$dS = \frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$



For (x,y) near $(0,0)$,

$$\|\vec{N}(x,y)\| = \frac{R}{\sqrt{R^2-x^2-y^2}} \simeq 1$$

The sphere S is almost flat and parallel to the xy plane for (x,y) near $(0,0)$. As such, we expect $\overset{(1)}{dS}$ and $\overset{(1)}{dx\,dy}$ to be similar in value

On the other hand, for (x,y) near the circle $x^2+y^2=R^2$, the denominator $\sqrt{R^2-x^2-y^2}$ is very small, implying that the

Jacobian $\dfrac{R}{\sqrt{R^2-x^2-y^2}}$ is very large in value. Near the outer boundary, the surface of the sphere is very steep and the element $dS^{(2)}$ will be very large in comparison with $dx\,dy^{(2)}$

Let's now compute the surface area of the hemisphere. (Of course, we know the answer: $A = 2\pi R^2$)

$$\iint_D \|\vec{N}(x,y)\|\,dx\,dy = \iint_D \frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

It's convenient to employ planar polar coordinates:

$$x = r\cos\theta,\quad y = r\sin\theta \qquad 0 \le r \le R \qquad 0 \le \theta \le 2\pi$$

$$dA = dx\,dy = r\,dr\,d\theta$$

Then

$$\iint_D \frac{R}{\sqrt{R^2-x^2-y^2}} = R\int_0^R\int_0^{2\pi} \frac{r}{\sqrt{R^2-r^2}}\,dr\,d\theta$$

$$= 2\pi R\int_0^R \frac{r}{\sqrt{R^2-r^2}}\,dr$$

$$= 2\pi R\left[-\sqrt{R^2-r^2}\right]_0^R$$

$$= 2\pi R^2 \checkmark$$

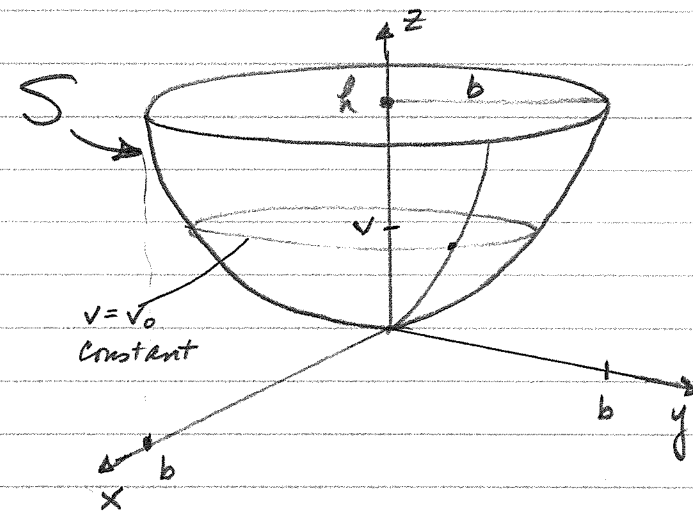
Example 5: INTEGRATION OF A SURFACE $z = f(x,y)$ USING TWO DIFFERENT PARAMETRIZATIONS

NOT DONE IN CLASS BUT PROVIDED AS SUPPLEMENTARY READING MATERIAL

The paraboloid of revolution about z-axis with height h and "base" radius b

Equation of this surface S:

$$\boxed{z = \left(\frac{x^2}{b^2} + \frac{y^2}{b^2}\right) h}$$



METHOD NO. 1

One possible parametrization

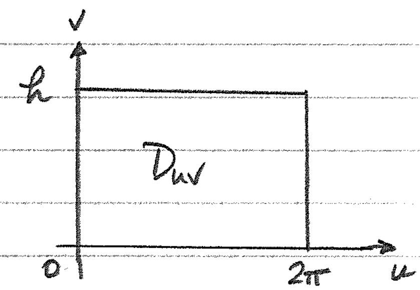
$0 \le z \le h$

let $v = z \qquad 0 \le v \le h$

Fixing $v \in [0,h]$ produces a circle

$$v = \left(\frac{x^2}{b^2} + \frac{y^2}{b^2}\right) h \Rightarrow \underbrace{\frac{h}{vb^2}x^2}_{\cos^2 u} + \underbrace{\frac{h}{vb^2}y^2}_{\sin^2 u} = 1$$

This suggests $\quad x = b\sqrt{\frac{v}{h}}\cos u \qquad y = b\sqrt{\frac{v}{h}}\sin u \qquad 0 \le u \le 2\pi$



Resulting parametrization

$$\vec{g}(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$= \left(\frac{b}{\sqrt{h}}\sqrt{v}\cos u,\ \frac{b}{\sqrt{h}}\sqrt{v}\sin u,\ v\right)$$

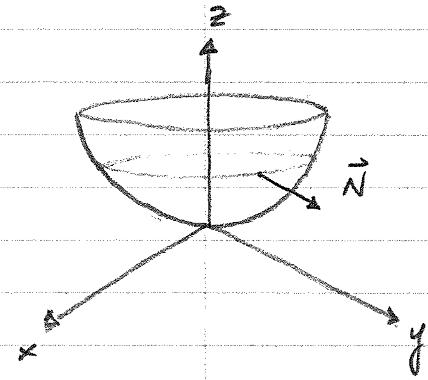
Tangent vectors

$$\vec{T}_u = \frac{\partial \vec{g}}{\partial u} = \left(-\frac{b}{\sqrt{h}}\sqrt{v}\sin u,\ \frac{b}{\sqrt{h}}\sqrt{v}\cos u,\ 0\right)$$

$$\vec{T}_v = \frac{\partial \vec{g}}{\partial v} = \left(\frac{1}{2}\frac{b}{\sqrt{h}}\frac{1}{\sqrt{v}}\cos u,\ \frac{1}{2}\frac{b}{\sqrt{h}}\frac{1}{\sqrt{v}}\cos u,\ 1\right)$$

A Normal vector

$$\vec{N} = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\frac{b}{\sqrt{h}}\sqrt{v}\sin u & \frac{b}{\sqrt{h}}\sqrt{v}\cos u & 0 \\ \frac{1}{2}\frac{b}{\sqrt{h}}\frac{1}{\sqrt{v}}\cos u & \frac{1}{2}\frac{b}{\sqrt{h}}\frac{1}{\sqrt{v}}\sin u & 1 \end{vmatrix}$$



$$= \Big(\underbrace{\frac{b}{\sqrt{h}}\sqrt{v}\cos u, \ \frac{b}{\sqrt{h}}\sqrt{v}\sin u}_{\text{outward from z axis}}, \underbrace{-\frac{1}{2}\frac{b^2}{h}}_{\text{points "downward"}}\Big)$$

Jacobian:

$$\| \vec{T}_u \times \vec{T}_v \| = \left[\frac{b^2}{h} v \cos^2 u + \frac{b^2}{h} v \sin^2 u + \frac{1}{4}\frac{b^4}{h^2} \right]^{1/2}$$

$$= \frac{b^2}{h}\left[\frac{h}{b^2} v + \frac{1}{4} \right]^{1/2}$$

$\Rightarrow$ Integration of a function $f(u,v)$ on S

$$\boxed{\int_S f\, dS = \int_{D_{uv}} f(u,v)\, \frac{b^2}{h}\left[\frac{h}{b^2}v + \frac{1}{4}\right]^{1/2} dA \ \begin{pmatrix} du\,dv \\ dv\,du \end{pmatrix}}$$

Example 5(a): Computation of surface area $A(S)$

Here, $f(u,v) = 1$

$$A(S) = \frac{b^2}{h} \iint \left[\frac{h}{b^2} v + \frac{1}{4} \right]^{1/2} dA$$

$$= \frac{b^2}{h} \underbrace{\left[\int_0^{2\pi} du \right]}_{2\pi} \left[\int_0^h \left[\frac{h}{b^2} v + \frac{1}{4} \right]^{1/2} dv \right]$$

$$\int_0^h \left[\frac{h}{b^2}v+\frac{1}{4}\right]^{1/2} dv = \frac{b^2}{h}\,\frac{2}{3}\left[\frac{h}{b^2}v+\frac{1}{4}\right]^{3/2}\Bigg|_0^h$$

$$= \frac{2}{3}\frac{b^2}{h}\left[\left(\frac{h^2}{b^2}+\frac{1}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right]$$

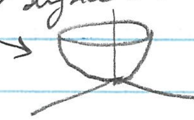
$$\Rightarrow A(S) = \frac{b^2}{h}\cdot 2\pi\cdot\frac{2}{3}\frac{b^2}{h}\left[\left(\frac{h^2}{b^2}+\frac{1}{4}\right)^{3/2}-\frac{1}{8}\right]$$

$$= \frac{4\pi}{3}\frac{b^4}{h^2}\left[\left(\frac{h^2}{b^2}+\frac{1}{4}\right)^{3/2}-\frac{1}{8}\right] = \frac{\pi}{6}\frac{b}{h^2}\left[\left(4h^2+b^2\right)^{3/2}-b^3\right]$$

Special case: $h=b=1$

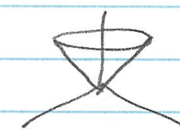
greater surface area ✓

$$A = \frac{\pi}{6}\left[5^{3/2}-1\right] \cong 5.33$$



Compare with cone, $h=b=1$

$$A = \pi b\sqrt{b^2+h^2} = \sqrt{2}\,\pi \simeq 4.44$$



Example 5(b) Computation of centroid (center of mass for a homogeneous surface $\rho=\rho_0$ constant

By symmetry $\bar{x}=\bar{y}=0$

$$\bar{z} = \frac{\iint_S z\,dS}{\iint_S dS} = \frac{1}{A(S)}\iint_S z\,dS$$

To compute $\iint z\,dS$, we must express $z = z(u,v)$

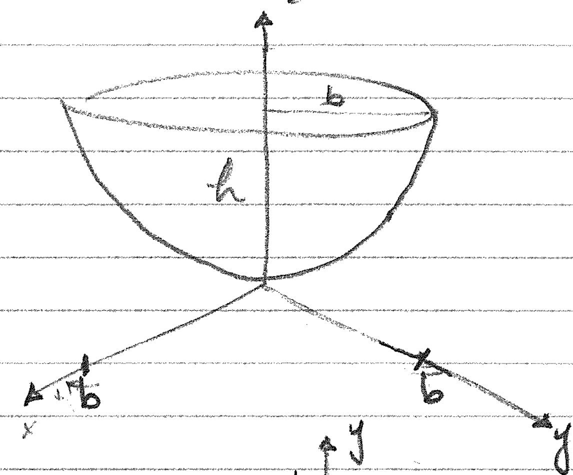
Here, $z(u,v) = v \quad 0 \le v \le h$

$$\iint_S z\,dS = \frac{b^2}{h}\iint_{D_{uv}} v\left[\frac{h}{b^2}v+\frac{1}{4}\right]^{1/2} dA$$

$\vdots$

METHOD NO. 2

PARAMETRIZATION OF $z = f(x,y)$ USING x & y as parameters

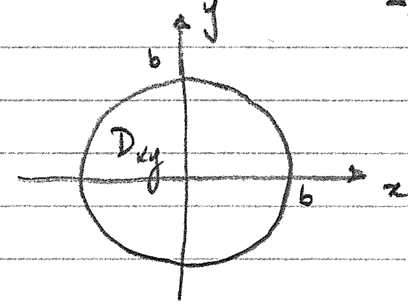


Paraboloid of revolution

$$z = \left(\frac{x^2}{b^2} + \frac{y^2}{b^2}\right) h$$

$$= \frac{h}{b^2}(x^2 + y^2)$$

$$= f(x,y)$$



Recall: Normal vector

$$\vec{N} = \vec{T}_x \times \vec{T}_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

$$= \left(-\frac{2h}{b^2}x, -\frac{2h}{b^2}y, 1\right)$$

This is actually the inward normal, but it doesn't matter.

Jacobian

$$\|\vec{N}\| = \|\vec{T}_x \times \vec{T}_y\|$$

$$= \left[\frac{4h^2}{b^4}x^2 + \frac{4h^2}{b^4}y^2 + 1\right]^{1/2}$$

Surface area

$$A(S) = \iint_{D_{xy}} \|\vec{N}(x,y)\|\, dx\, dy$$

$$= \iint \left[\frac{4h^2}{b^4}x^2 + \frac{4h^2}{b^4}y^2 + 1\right]^{1/2} dA$$

$dx\,dy$ or $dy\,dx$

Easier to use polar coordinates $0 \le r \le b$

$0 \le \theta \le 2\pi$

$$dx\,dy = r\,dr\,d\theta$$

$$A(S) = \int_0^{2\pi} \int_0^{b} \left[\frac{4h^2}{b^4} r^2 + 1 \right]^{1/2} r\,dr\,d\theta$$

$$= 2\pi \int_0^{b} \left[\frac{4h^2}{b^4} r^2 + 1 \right]^{1/2} r\,dr$$

$$= 2\pi \cdot \frac{1}{2} \cdot \frac{2}{3} \frac{b^4}{4h^2} \left[\frac{4h^2}{b^4} r^2 + 1 \right]^{3/2} \Bigg|_0^{b}$$

$$= \frac{\pi}{6} \frac{b^4}{4h^2} \left[\left(\frac{4h^2}{b^2} + 1 \right)^{3/2} - 1 \right]$$

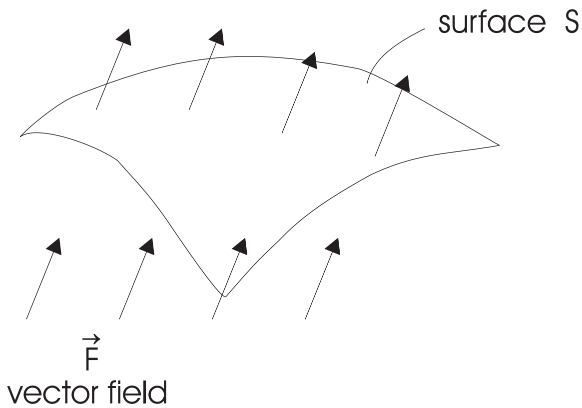
To compare with previous result (Page 19.7)

$$= \frac{\pi}{6} \frac{b}{h^2} \left[\left(4h^2 + b^2 \right)^{3/2} - b^3 \right] \checkmark$$

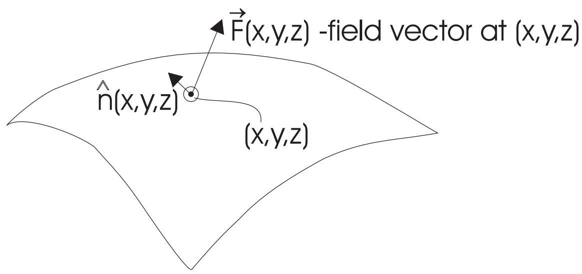
Surface integrals (cont'd)

Surface integrals of vector functions and "flux"

Relevant section from AMATH 231 Course Notes: 3.2.2, p. 77



We return to the concept of the "flux" of a vector field $\mathbf{F}$. Now, however, we are concerned with the flux of $\mathbf{F}$ through s surface S.



The unit normal $\hat{\mathbf{N}}$ to S at (x, y, z) is usually directed "outward" (as dictated by the physical problem).

$(x, y, z) \in S$ is centered in an infinitesimal element dS of area on S.

The "amount" of $\mathbf{F}$ pointing in the direction of $\hat{\mathbf{N}}$ is the projection of $\mathbf{F}$ in the direction of $\hat{\mathbf{N}}$, i.e., $\mathbf{F} \cdot \hat{\mathbf{N}}$. The (infinitesimal) flux of $\mathbf{F}$ through the surface element dS is $\mathbf{F} \cdot \hat{\mathbf{N}}\, dS$. To obtain the total flux through the surface S, we must integrate over all elements dS centered at $(x, y, z) \in S$. We denote this vector surface integral as follows

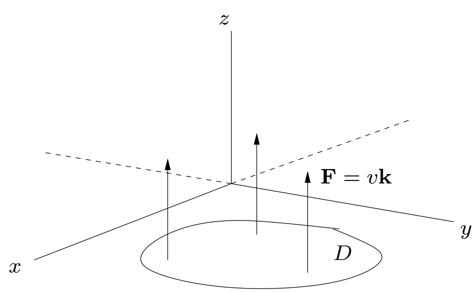
$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS. \tag{11}$$

In order to understand this idea better, let us examine a particular physical application of the flux integral.

Flux in terms of fluid flow

First, consider a region D that lies in the xy-plane as sketched below. Suppose that a fluid is passing through this region. For the moment, we assume that motion of the fluid is perpendicular to region D, travelling in the direction of the positive z-axis. Moreover, we assume that the speed of the fluid particles crossing D is constant throughout the region. As such, we are assuming that the velocity field of the fluid is

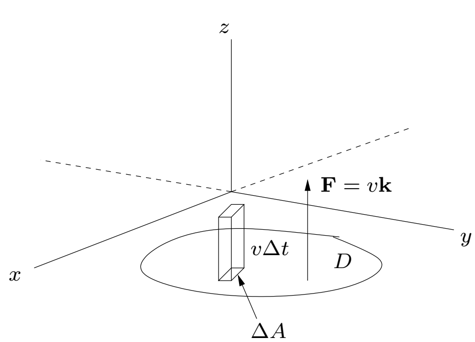
$$\mathbf{F} = v\mathbf{k}, \qquad v > 0 \text{ (constant)} . \tag{12}$$



We first ask the question: How much fluid flows through region D during a time interval Δt? Consider a tiny rectangular element of area $\Delta A = \Delta x \Delta y$ centered at a point (x, y) in D. After a time Δt, the fluid particles situated in this element will have moved a distance $v\Delta t$ upward. The volume of fluid that has passed through this element ΔA on D is the volume of the box of base area ΔA and height $v\Delta t$:

$$v\Delta t \Delta A. \tag{13}$$

This box is sketched below.



The total volume ΔV of fluid that has passed through region D over the time interval Δt is obtained by summing up over all area elements ΔA in D:

$$\Delta V = v\Delta t \int\int_D dA = v\Delta t A(D), \tag{14}$$

where $A(D)$ denotes the area of D. Of course, this is a rather trivial result: the volume of fluid passing through D is simply the volume of the solid of base area $A(D)$ and height $v\Delta t$. Dividing both sides by Δt, we have

$$\frac{\Delta V}{\Delta t} = vA(D). \tag{15}$$

In the limit $\Delta t \to 0$, we have the instantaneous rate of change of the volume of fluid passing through region D, or simply the *rate of fluid flow* through region D:

$$V'(t) = vA(D). \tag{16}$$

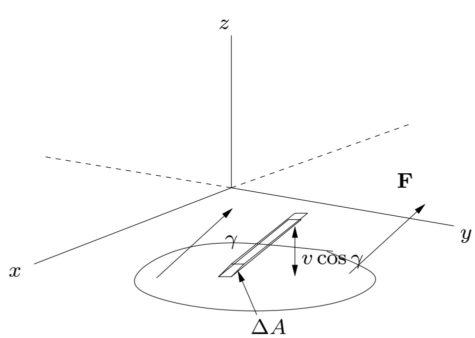
This quantity is the *flux* of the vector field $\mathbf{v}$ through region D.

Now suppose that the fluid is now moving at a constant speed v through region D but not necessarily at right angles to it, i.e., not necessarily parallel to its normal vector $\mathbf{k}$. We shall suppose that

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \qquad \| \mathbf{v} \| = v \tag{17}$$

and let γ denote the angle between $\mathbf{v}$ and the normal vector $\mathbf{k}$.

In this case, the fluid particles that pass through the tiny element ΔA after a time interval Δt form a parallelopiped of base area ΔA and height $v\cos\gamma\Delta t$, as sketched below.



The volume of this box is

$$v\cos\gamma\Delta t\Delta A. \tag{18}$$

(Think of this tower of fluid as a deck of playing cards that has been somewhat sheared. When you slide the cards back to form a rectangular arrangement, the height of the deck is $v\delta t\cos\gamma$.)

The total volume ΔV of fluid that has passed through region D over the time interval Δt is obtained by summing up over all area elements ΔA in D:

$$\Delta V = v \cos\gamma \Delta t \int\int_D dA = v\cos\gamma \Delta t A(D), \tag{19}$$

Dividing both sides by Δt, we have

$$\frac{\Delta V}{\Delta t} = v\cos\gamma A(D). \tag{20}$$

In the limit $\Delta t \to 0$, we obtain the flux of the vector field $\mathbf{v}$ through region D:

$$V'(t) = v\cos\gamma A(D). \tag{21}$$

We shall rewrite this flux as follows,

$$V'(t) = \mathbf{v} \cdot \hat{\mathbf{N}}\ A(D), \tag{22}$$

since $\hat{\mathbf{N}} = \mathbf{k}$ is the normal vector to surface D. Note that this general case includes the first case, $\gamma = 0$. And in the case that $\gamma = \pi/2$, there is no flow through the region D, so the flux is zero.

Of course, the above results have been rather trivially obtained since (i) the vector fields are constant and (ii) the region D is flat. Let us now generalize the first case, i.e., the vector field $\mathbf{v}$ is assumed to be *nonconstant* over region D, i.e.,

$$\mathbf{v}(x,y) = v_1(x,y)\mathbf{i} + v_2(x,y)\mathbf{j} + v_3(x,y)\mathbf{k}. \tag{23}$$

In this case, the total volume ΔV of fluid that has passed through region D over the time interval Δt is obtained by summing up over all area elements ΔA in D:

$$\Delta V = \Delta t \int\int_D \mathbf{v}(x,y)\cdot\hat{\mathbf{N}}\ dA = \Delta t \int\int_D v_3(x,y)\ dA. \tag{24}$$

Once again dividing by Δt and taking the limit $\Delta t \to 0$, we obtain the total flux of $\mathbf{v}$ through region D:

$$V'(t) = \int\int_D \mathbf{v}(x,y)\cdot\hat{\mathbf{N}}\ dA \tag{25}$$

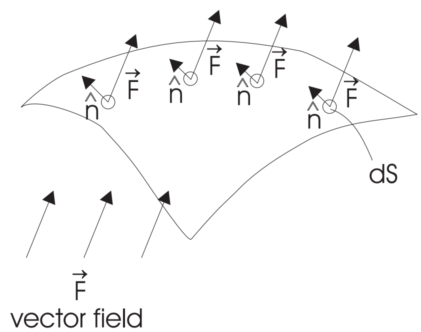
Now suppose that we were concerned with rate of *mass flow* through region D. The amounts/volumes of fluid examined earlier would be replaced by amounts of mass flowing through a surface element. This means replacing the velocity vector field $\mathbf{v}$ by the *momentum field* $\mathbf{F} = \rho\,\mathbf{v}$, where ρ is the mass density. The rate of transport of mass through region D would then be given by

$$M'(t) = \int\int_D \mathbf{F}(x,y)\cdot\hat{\mathbf{N}}\ dA = \int\int_D \rho(x,y)\ \mathbf{v}(x,y)\cdot\hat{\mathbf{N}}\ dA \tag{26}$$

This concludes our discussion of this simple problem involving fluid flow through a *flat* surface.

Generalization to arbitrary surfaces

We now wish to generalize the above result to general surfaces in $\mathbf{R^3}$. In other words, we do not require the surface to be flat, as was region D in the plane, but rather a general surface S in $\mathbf{R^3}$ – for example, a portion of a sphere, or perhaps the entire sphere. In the "spirit of calculus," we divide the surface S into tiny infinitesimal pieces dS. We then construct a normal vector $\hat{\mathbf{N}}$ to each surface element dS at a point in dS, as sketched below.



We then form the dot product of the vector field $\mathbf{F}$ at that point with the normal vector $\hat{\mathbf{N}}$. This will represent the local flux of $\mathbf{F}$ through the surface element dS. To obtain the total flux through the surface S, we add up the fluxes of all elements dS – an integration over S that is denoted as

$$\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS. \tag{27}$$

This is the "flux" of $\mathbf{F}$ through surface S. Note that in some books, especially Physics books, the vector surface integral is denoted as

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}. \tag{28}$$

Here, the infinitesimal surface area element is a *vector* that is defined as

$$d\mathbf{S} = \hat{\mathbf{N}}\, dS, \tag{29}$$

where dS is the infinitesimal surface element and $\hat{\mathbf{N}}$ is the unit normal vector to the surface element.

In other books, the infinitesimal surface element is denoted as $d\mathbf{A} = \hat{\mathbf{N}}dS$, so that the flux integral is denoted as

$$\int\int_S \mathbf{F} \cdot d\mathbf{A}. \tag{30}$$

Practical computation of flux integrals

The integrand in a vector surface integral, $\mathbf{F} \cdot \hat{\mathbf{N}}$, will generally depend on the x, y, z. Since these coordinates are restrcted to the surface S of interest, they, hence the integrand, will depend on u and v (i.e., $(x(u,v), y(u,v), z(u,v))$).

As we saw in the introduction to this section, in some special cases that occur in physics, e.g. gravity, electricity/magnetism, the vector fields and the surfaces are spherically symmetric. In such cases, we can compute fluxes in a rather straightforward way using the definition of the flux integral.

In general, however, the computation of flux integrals may not be as straightforward as for these elementary examples. As in the case of surface integrals for scalar functions, we'll have to use the parametrization of the surface.

Notice that in the total flux integral,

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

the integrand $\mathbf{F} \cdot \hat{\mathbf{N}}$ is a *scalar*–call it $f(x, y, z)$ – so that the flux integral becomes

$$= \int\int_S f\, dS$$

But we know how to compute surface integrals of scalar functions! If we parametrize the surface S, $(\mathbf{g}(u,v), (u,v) \in D_{uv})$, then

$$\int\int_S f\, dS = \int\int_{D_{uv}} f(\mathbf{g}(u,v)) \|\mathbf{N}(u,v)\| dA \qquad (31)$$

Here, f is the dot product $\mathbf{F} \cdot \hat{\mathbf{N}}$ on surface S:

$$f(\mathbf{g}(u,v)) = \underbrace{\mathbf{F}(\mathbf{g}(u,v))}_{F\ on\ surface} \cdot \underbrace{\hat{\mathbf{N}}(u,v)}_{\substack{appropriate\\ unit\\ normal\ on\\ S}}$$

What is $\hat{\mathbf{N}}$?

$$\hat{\mathbf{N}}(u,v) = \frac{\mathbf{N}(u,v)}{\|\mathbf{N}(u,v)\|}$$

So Equation (31) becomes

$$= \int\int_S \mathbf{F}(\mathbf{g}(u,v)) \cdot \frac{\mathbf{N}(u,v)}{\|\mathbf{N}(u,v)\|} \|\mathbf{N}(u,v)\| dA$$

The final result is:

$$\boxed{\int\int_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \int\int_{D_{uv}} \mathbf{F}(\mathbf{g}(u,v))\cdot\mathbf{N}(u,v)\,dA}$$

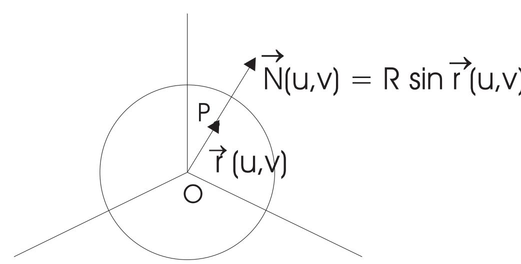
The vector surface integral is translated into a surface integral involving the *scalar function* $\mathbf{F}\cdot\hat{\mathbf{N}}$. The integration on the right-hand side is performed over the region in uv parameter space, D_{uv}, that generates the surface S.

Some of the examples which are presented below will involve either spherical surface S_R – either a part of it or its entirety – of radius R and centered at $(0,0,0)$. Recall that the parametrization of this surface is given by

$$\mathbf{g}(u,v) = (R\sin v\cos u, R\sin v\sin u, R\cos v), \qquad 0\le u\le 2\pi, \quad 0\le v\le \pi. \tag{32}$$

Also recall from our earlier work that the outward normal associated with this parametrization was given by

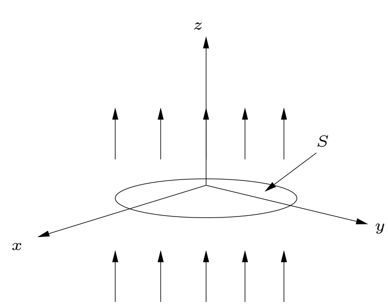
$$\mathbf{N} = \mathbf{T}_v\times\mathbf{T}_u = R\sin v\,\mathbf{g}(u,v). \tag{33}$$



1. As a kind of warmup, let us first compute the total flux of the vector field $\mathbf{F} = \mathbf{k} = (0,0,1)$ through the circular planar region S on the xy-plane,

$$S = \{(x,y,0)\mid x^2+y^2\le R^2\}. \tag{34}$$

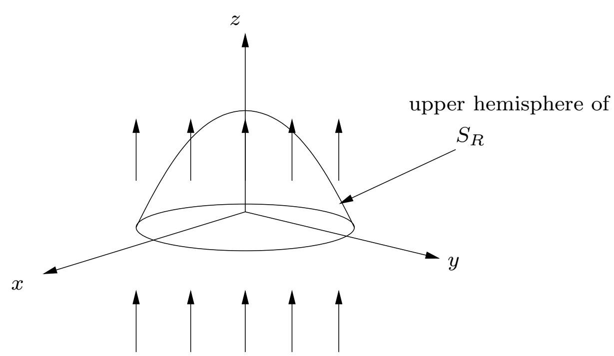
The boundary of this region is the intersection of the spherical surface S_R with the xy-plane. The situation is sketched below.



The computation of the flux of $\mathbf{F}$ through this region is quite straightforward since the surface is a plane and the vector field is constant. In fact, the surface lies on the xy-plane which implies that its unit normal vector $\hat{N} = \mathbf{k}$. We can simply use Eq. (21) from the earlier part of this lecture:

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dA = \int\int_S (0,0,1) \cdot (0,0,1)\, dA = \int\int_S dA = \pi R^2 . \tag{35}$$

2. Now let us compute the total flux of the same vector field, $\mathbf{F} = \mathbf{k} = (0,0,1)$, but through the upper hemispherical surface of S_R, as sketched below.



We must compute the surface flux integral,

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \int\int_{D_{uv}} \mathbf{F}(\mathbf{g}(u,v)) \cdot \mathbf{N}(u,v)\, dA$$

The integrand must be evaluated over the surface S. Here, $\mathbf{F}(\mathbf{g}(u,v)) = (0,0,1)$ so that

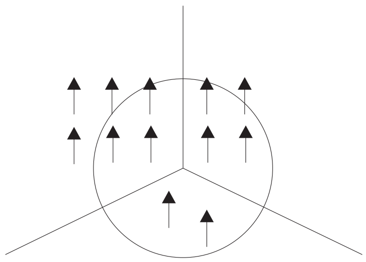
$$\begin{aligned} \mathbf{F}(\mathbf{g}(u,v)) \cdot \mathbf{N}(u,v) &= (0,0,1) \cdot R\sin v(R\sin v\cos u, R\sin v\sin u, R\cos v) \\ &= R^2 \sin v \cos v. \end{aligned}$$

Now integrate over surface, keeping in mind that we're integrating only over the upper hemisphere, so that $0 \le v \le \frac{1}{2}\pi$:

$$\begin{aligned} \int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS &= \int_0^{\pi/2} \int_0^{2\pi} R^2 \sin v \cos v\, du dv \\ &= 2\pi R^2 \int_0^{\pi/2} \sin v \cos v\, dv \\ &= (2\pi R^2)\, \underbrace{\frac{1}{2} \sin^2 v \Big|_0^{\frac{1}{2}\pi}} \\ &= \pi R^2 . \end{aligned}$$

Note that we obtain the same result as in Example 1. The net flux of the fluid passing through the hemispherical surface is the same as the flux of the fluid passing through the projection of this surface onto the xy-plane.

3. Let us now compute the **total outward flux** of the vector field $\mathbf{F} = \mathbf{k} = (0, 0, 1)$ of Examples 1 and 2 through the **entire spherical surface** S_R.



It appears that the amount of vector field entering the sphere from the bottom is equal to the amount that is exiting at the top. One would conjecture, then, that the total flux across the sphere is zero.

Once again, we must compute the flux integral,

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \int\int_{D_{uv}} \mathbf{F}(\mathbf{g}(u, v)) \cdot \mathbf{N}(u, v)\, dA$$

The integrand must be evaluated over the surface. Here, $\mathbf{F}(\mathbf{g}(u, v)) = (0, 0, 1)$, so that

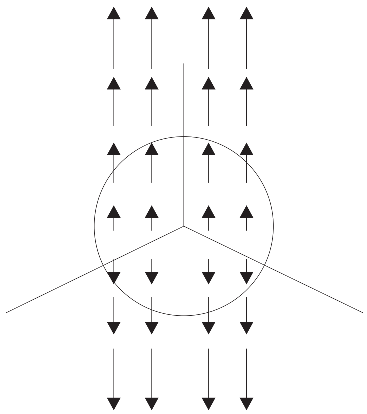
$$\begin{aligned} \mathbf{F}(\mathbf{g}(u, v)) \cdot \mathbf{N}(u, v) &= (0, 0, 1) \cdot R \sin v (R \sin v \cos u, R \sin v \sin u, R \cos v) \\ &= R^2 \sin v \cos v. \end{aligned}$$

Now integrate over entire spherical surface: In this case $0 \leq v \leq \pi$:

$$\begin{aligned} \int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS &= \int_0^{\pi} \int_0^{2\pi} R^2 \sin v \cos v\, du dv \\ &= 2\pi R^2 \int_0^{\pi} \sin v \cos v\, dv \\ &= (2\pi R^2)\, \underbrace{\frac{1}{2} \sin^2 v \Big|_0^{\pi}}_{=0} \\ &= 0\,. \end{aligned}$$

This was as expected – net outward flow through the top is equal to the net inward flow through the bottom.

4. Compute the total outward flux of the vector field $\mathbf{F} = z\mathbf{k} = (0, 0, z)$ through the spherical surface S_R.



The arrows in the vector field get longer as you move away from the xy-plane. Moreover, for $z > 0$, they point upward and for $z < 0$, they point downward. Therefore, there is a net outward flow from the xy-plane through the surface. We expect a nonzero, in fact, positive outward flux here.

On the surface,

$$\begin{aligned}\mathbf{F}(\mathbf{g}(u,v)) &= (0,0,z(u,v)) \\ &= (0,0,R\cos v),\end{aligned}$$

so that

$$\begin{aligned}\mathbf{F}(\mathbf{g}(u,v)) \cdot \mathbf{N}(u,v) &= (0,0,R\cos v) \cdot R\sin v(R\sin v\cos u, R\sin v\sin u, R\cos v) \\ &= R^3\cos^2 v\sin v.\end{aligned}$$

The flux is therefore given by

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS &= \int_0^\pi\int_0^{2\pi} R^3\cos^2 v\sin v\,dudv \\ &= 2\pi R^3\int_0^\pi \cos^2 v\sin v\,dv \\ &= (2\pi R^3)\left(-\frac{1}{3}\cos^3 v\Bigg|_0^\pi\right) \\ &= (2\pi R^3)(\frac{2}{3}) \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

By symmetry, you will find the same result for $\mathbf{F} = x\mathbf{i}$ or $\mathbf{F} = y\mathbf{j}$ (i.e., same type of vector field, just in a different direction).

Note: The fact that the total flux for these vector fields is equal to the volume of the sphere is **not** a coincidence, as we'll see in the next lecture. It is a consequence of the celebrated *Divergence Theorem.*

Gauss Divergence Theorem in $\mathbf{R}^3$

`Relevant section of AMATH 231 Course Notes:  Section 4.2.1`

In this section, we are concerned with the outward flux of a vector field (through a smooth/piecewise smooth) surface S that encloses a region $V \subset \mathbf{R}^3$. Typically, in physics such surfaces are spheres, boxes, parallelpipeds or cylinders. This is the subject of the celebrated **Gauss Divergence Theorem**, the three-dimensional version of the Divergence Theorem in the Plane of a previous lecture.

The Gauss Divergence Theorem is one of the most important results of vector calculus. It is not as important for computational purposes as for conceptual developments. It provides the basis for the important equations in electromagnetism (Maxwell's equations), fluid mechanics (continuity equation) and continuum mechanics in general (heat equation, diffusion equation).

It is sufficient to consider a somewhat simplified version of the general Divergence Theorem.

A simplified version of the Divergence Theorem:

Let S be a "nice" (i.e., piecewise smooth) closed and nonintersecting surface that encloses a region $D \subset \mathbf{R}^3$, such that an outward unit normal vector $\hat{\mathbf{N}}$ exists at all points on S. Also assume that a vector field $\mathbf{F}$ and its derivatives are defined over region D and its boundary S.

The Divergence Theorem states that:

$$\underbrace{\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS}_{surface\ integral} = \underbrace{\int\int\int_D \operatorname{div} \mathbf{F}\, dV}_{volume\ integral}. \tag{36}$$

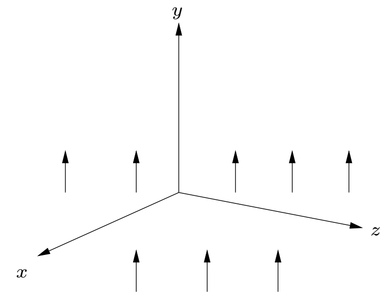
Once again, we have assumed that $\operatorname{div} \mathbf{F}$ exists at all points in V.

You have already seen a version of this theorem – the two-dimensional Divergence Theorem in the plane. It expressed the total outward flux of a 2D vector field $\mathbf{F}$ through a closed curve C in the plane as an integral of the divergence of $\mathbf{F}$ over the region D enclosed by C:

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{N}}\, ds = \int\int_D \operatorname{div} \mathbf{F}\, dA. \tag{37}$$

Examples: In what follows, unless otherwise indicated, the surface S is an arbitrary surface in $\mathbf{R}^3$ satisfying the conditions of the Divergence Theorem.

1. The vector field $\mathbf{F} = \mathbf{k} = (0, 0, 1)$. This vector field could be viewed as the velocity field of a fluid that is travelling with constant speed in the positive z-direction:



The divergence of this vector field is zero:

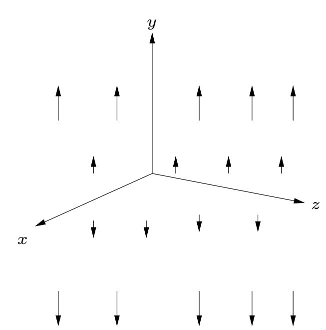
$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0. \tag{38}$$

More importantly, it exists at all points in $\mathbf{R}^3$, i.e., there are no singularities, so that we may employ the Divergence Theorem. Therefore, for any surface S enclosing a region D, we have

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \int\int\int_D \text{div}\,\mathbf{F}\, dV = \int\int\int_D 0\ dV = 0. \tag{39}$$

In other words, the total outward flux of $\mathbf{F}$ over the surface S is zero. In terms of the fluid analogy, fluid is entering the region through surface S from the bottom at the same rate that it is leaving it at the top. There is no creation of extra fluid anywhere inside region D that would cause a nonzero flux.

2. The vector field $\mathbf{F} = z\mathbf{k} = (0, 0, z)$. A sketch of the vector field is given below.



This field could be viewed as the velocity field of a liquid that originates from the xy-plane and travels upward and downward away from it. As it moves away, it accelerates, since the velocity is proportional to the distance from the xy-plane.

The divergence of this field is

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 1. \tag{40}$$

Once again, the divergence exists at all points in $\mathbf{R}^3$. Therefore, by the Divergence Theorem

$$\int\int_{S_R} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \int\int\int_D \text{div}\, \mathbf{F}\, dV = \int\int\int_D 1\ dV = V(D), \tag{41}$$

the volume of region D.

Note that the same result for the flux, i.e., Eq. (41), would be obtained for the following vector fields:

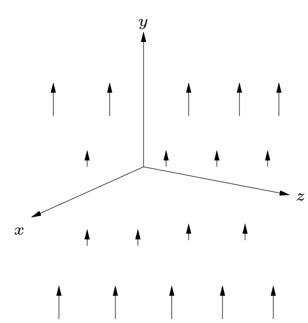
$$(i) \quad \mathbf{F} = x\mathbf{i}, \qquad (ii) \quad \mathbf{F} = y\mathbf{j}, \tag{42}$$

since the divergence of each of these vector fields is 1. And the list does not stop here. Consider the set of all vector fields of the form

$$\mathbf{F} = c_1 x\mathbf{i} + c_2 y\mathbf{j} + c_3 z\mathbf{k}, \quad \text{where} \ \ c_1 + c_2 + c_3 = 1. \tag{43}$$

In all cases, we have div $\mathbf{F} = 1$, so that the total outward flux of each of these fields through the surface S is $V(D)$, the volume of region D.

3. The vector field $\mathbf{F} = z^2\mathbf{k} = (0, 0, z^2)$. Here, all arrows of $\mathbf{F}$ point upward, as sketched below.



This could be visualized as fluid that emanates from the xy-plane to travel upward, accelerating as it moves away from the plane, along with fluid that approaches the xy-plane from below, decelerating as it gets closer.

The divergence of $\mathbf{F}$ is

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z^2) = 2z \tag{44}$$

Therefore, by the Divergence Theorem

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = \int\int\int_D \operatorname{div}\mathbf{F}\,dV = 2\int\int\int_D \; z\;dV. \tag{45}$$

The value of this integral will depend on the region D. In principle, if we knew the region, we could integrate over it, using the techniques for integration in $\mathbf{R}^3$ developed earlier in the course.

There is one interesting point regarding this integral: It is related to the z coordinate of the centroid of region D. Recall that

$$\bar{z} = \frac{\int\int\int_D z\;dV}{\int\int\int_D \;dV}. = \frac{\int\int\int_D z\;dV}{V(D)}, \tag{46}$$

implying that

$$\int\int\int_D \; z\;dV = V(D)\bar{z}. \tag{47}$$

Therefore, Eq. (45) becomes

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS = 2\bar{z}V(D). \tag{48}$$

Note that if the surface S is located in the upper half-plane, i.e., $z > 0$, then $\bar{z} > 0$, implying that the total outward flux is positive. However, if the surface S is located in the lower half-plane, i.e., $z < 0$, the total outward flux is negative. Why is this so? And why would the total outward flux be directly proportional to the volume $V(D)$ of the region D enclosed by the surface S?