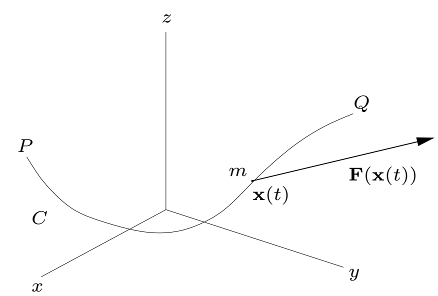
Lecture 11

Line integrals of vector-valued functions (cont'd)

In the previous lecture, we considered the following physical situation: A force, $\mathbf{F}(\mathbf{x})$, which is not necessarily constant in space, is acting on a mass m, as the mass moves along a curve C from point P to point Q as shown in the diagram below.



The goal is to compute the total amount of work W done by the force. Clearly the constant force/straight line displacement formula,

$$W = \mathbf{F} \cdot \mathbf{d}, \tag{1}$$

where $\mathbf{F}$ is force and $\mathbf{d}$ is the displacement vector, does not apply here. But the fundamental idea, in the "Spirit of Calculus," is to break up the motion into tiny pieces over which we can use Eq. (1 as an approximation over small pieces of the curve. We then "sum up," i.e., integrate, over all contributions to obtain W. The total work is the **line integral of the vector field F over the curve C:**

$$W = \int_C \mathbf{F} \cdot d\mathbf{x}. \tag{2}$$

Here, we summarize the main steps involved in the computation of this line integral:

1. **Step 1: Parametrize the curve** C We assume that the curve C can be parametrized, i.e.,

$$\mathbf{x}(t) = \mathbf{g}(t) = (x(t), y(t), z(t)), \qquad t \in [a, b], \tag{3}$$

so that $\mathbf{g}(a)$ is point P and $\mathbf{g}(b)$ is point Q. From this parametrization we can compute the velocity vector,

$$\mathbf{v}(t) = \mathbf{g}'(t) = (x'(t), y'(t), z'(t)). \tag{4}$$

2. **Step 2: Compute field vector $\mathbf{F}(\mathbf{g}(t))$ over curve C**

$$\begin{aligned} \mathbf{F}(\mathbf{g}(t)) &= \mathbf{F}(x(t), y(t), z(t)) && (5) \\ &= (F_1(x(t), y(t), z(t),\ F_2(x(t), y(t), z(t),\ F_3(x(t), y(t), z(t)))\,, \quad t \in [a, b]\,. && (6) \end{aligned}$$

3. **Step 3: Construct the integrand of $\int_C \mathbf{F} \cdot d\mathbf{x}$, i.e., the dot product**

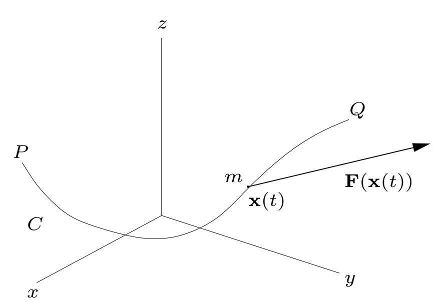
$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\,. \tag{7}$$

4. **Step 4: Compute line integral as a definite integral over parameter t**

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt\,. \tag{8}$$

Some important properties and ideas involving line integrals of vector fields

1. **Directionality:** A line integral of a vector field $\mathbf{F}$ over a curve C must involve a specific direction of travel over C. With reference to the figure employed earlier,



the work W done by the force $\mathbf{F}$ in moving mass m from point P to point Q was written as the line integral,

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt\,, \tag{9}$$

where it was understood that the parametrization $\mathbf{g}(t)$ of C was such that

$$\mathbf{g}(a) = P, \quad \mathbf{g}(b) = Q\,. \tag{10}$$

As such, it would have been better if the LHS of the above equation were written as

$$\int_{C_{PQ}} \mathbf{F} \cdot d\mathbf{x}\,, \tag{11}$$

where C_{PQ} denotes the curve C with path starting at P and ending at Q.

As is the case with Riemann integrals over the real interval $[a,b] \subset \mathbf{R}$, if you reverse the direction of the integration, you obtain the negative result, i.e.,

$$\int_{C_{QP}} \mathbf{F} \cdot d\mathbf{x} = -\int_{C_{PQ}} \mathbf{F} \cdot d\mathbf{x}\,, \tag{12}$$

where C_{QP} is the path over curve C that starts at Q and ends at P.

Proof: Given that $\mathbf{g}(t)$, $t \in [a,b]$, is the parametrization of curve C starting at P and ending at Q, i.e.,

$$\mathbf{g}(a) = P\,, \quad \mathbf{g}(b) = Q\,, \tag{13}$$

define the new parameter

$$\tau = a + b - t\,, \quad t \in [a,b]\,. \tag{14}$$

As t runs from a to b, the parameter τ runs from b to a, i.e., the reverse direction. We then define

$$\mathbf{h}(t) = \mathbf{g}(\tau) = \mathbf{g}(a+b-t)\,. \tag{15}$$

For all values $t \in [a,b]$, the parameter $\tau = a+b-t \in [a,b]$, which implies that all points $\mathbf{h}(t)$ lie on C. Clearly,

$$\mathbf{h}(a) = \mathbf{g}(b) = Q\,, \quad \mathbf{h}(b) = \mathbf{g}(a) = P\,. \tag{16}$$

In other words, the parametrization $\mathbf{h}(t)$ starts at Q and ends at P. From the Chain Rule,

$$\mathbf{h}'(t) = \frac{d}{dt}\mathbf{g}(\tau) = \frac{d}{d\tau}\mathbf{g}(\tau)\frac{d\tau}{dt} = -\mathbf{g}'(\tau)\,. \tag{17}$$

As expected, the velocity vectors at a given point $\mathbf{h}(t) = \mathbf{g}(\tau)$ on the curve point in opposite directions.

We now compute the line integral of the vector field $\mathbf{F}$ over the curve starting at Q and ending at P, which we denote as curve C_{QP} as follows,

$$\begin{aligned} \int_{C_{QP}} \mathbf{F} \cdot d\mathbf{x} &= \int_a^b \mathbf{F}(\mathbf{h}(t)) \cdot \mathbf{h}'(t)\,dt \\ &= \int_a^b \mathbf{F}(\mathbf{g}(a+b-t) \cdot \mathbf{g}'(a+b-t)\,dt \\ &= \int_b^a \mathbf{F}(\mathbf{g}(\tau) \cdot \mathbf{g}'(\tau)(-d\tau) \qquad \left(\text{since } \frac{dt}{d\tau} = -1\right) \end{aligned}$$

$$\begin{aligned} &= -\int_a^b \mathbf{F}(\mathbf{g}(\tau)\cdot\mathbf{g}(\tau)\,d\tau \\ &= -\int_{C_{AB}} \mathbf{F}\cdot d\mathbf{x}\,. \qquad (18) \end{aligned}$$

To get from the second-last line to the final line, we note that the parameter τ is being integrated from a to b. It doesn't matter what we call the parameter at this point.

Moral of the story: When dealing with the line integral $\int_C \mathbf{F}\cdot d\mathbf{x}$, over a curve C, we must also specify the **orientation** of the curve over which the integration is to be performed.

2. **Linearity:** From the Riemann sum definition of the line integral, it follows that

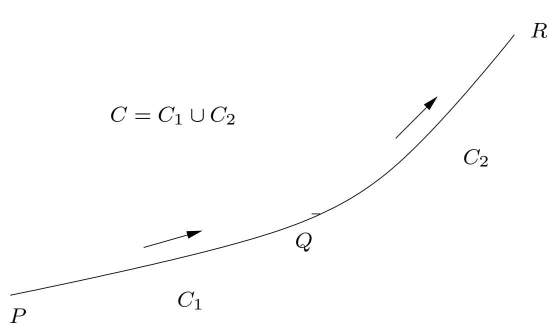
$$\begin{aligned} \int_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{x} &= \int_C \mathbf{F}\cdot d\mathbf{x} + \int_C \mathbf{G}\cdot d\mathbf{x} \\ \int_C (c\mathbf{F})\cdot d\mathbf{x} &= c\int_C \mathbf{F}\cdot d\mathbf{x}\,, \qquad (19) \end{aligned}$$

where $c \in \mathbf{R}$ is a constant scalar.

3. **Additivity over paths:** Let C be a C^1 curve. (This means that if $\mathbf{g}(t)$, $t \in [a,b]$, is a parametrization of C, the tangent vector $\mathbf{g}'(t)$ is continuous at all $t \in [a,b]$.) Furthermore, suppose that C may be expressed as a union of two curves C_1 and C_2 joined end-to-end and oriented consistently (i.e., orientations of C_1 and C_2 are compatable), in which case we may write

$$C = C_1 \cup C_2\,. \qquad (20)$$

This is sketched in the figure below. Then

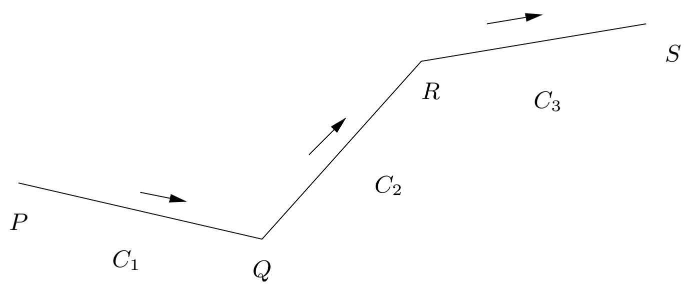


$$\int_C \mathbf{F}\cdot d\mathbf{x} = \int_{C_1} \mathbf{F}\cdot d\mathbf{x} + \int_{C_2} \mathbf{F}\cdot d\mathbf{x}\,. \qquad (21)$$

4. **Line integrals over piecewise C^1 curves:** Let C be a continuous curve which is piecewise C^1, i.e.,

$$C = C_1 \cup C_2 \cup \cdots \cup C_n \,, \tag{22}$$

where each of the individual pieces C_i, $1 \leq i \leq n$ is of class C^1. An example is sketched below.



Then the line integral of $\mathbf{F}$ over C is the sum of the line integrals of $\mathbf{F}$ over the C_i, i.e.,

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_{C_1} \mathbf{F} \cdot d\mathbf{x} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{x} \,. \tag{23}$$

Note that a consistent orientation of the curves C_i is once again assumed.

See Example 2.3 starting on Page 39 of the AMATH 231 Course Notes for a worked-out example.

5. **Region of integration:** For sufficiently "nice" vector fields $\mathbf{F}$, i.e., those whose partial derivatives not only exist but are continuous everywhere, we shall be able to consider/compute line integrals of the form

$$\int_C \mathbf{F} \cdot d\mathbf{x} \tag{24}$$

for arbitrary bounded curves in the appropriate space $\mathbf{R}^n$.

In many applications to Physics, however, vector fields of interest have "singularities", i.e., points at which either the vector field $\mathbf{F}$ is either discontinuous or has discontinuous partial derivatives. As such, it may be necessary to restrict the domain of integration.

In most treatments of line integrals, one usually specifies some basis property or properties of the domain $D \subset \mathbf{R}^n$ over which the line integration is to be performed. It is then assumed that any curve C over which the integration is performed belongs to this domain D. Two basic properties that will be assumed for the moment, unless otherwise specified are (i) openness and (ii) connectedness.

   (a) The set $D \subset \mathbf{R}^n$ is usually assumed to be an open set in $\mathbf{R}^n$, which may even include the entire set $\mathbf{R}^n$. When the set D is open, we don't have to worry about boundaries – given any point $x \in D$, one can always find an ϵ-neighbourhood of D centered at x for some $\epsilon > 0$. (Think of the difference between the open interval $(a, b) \subset \mathbf{R}$ and the closed interval $[a, b] \subset \mathbf{R}$.)

(b) The set $D \subset \mathbf{R}^n$ is assumed to be **connected**: Given **any** two distinct points $\mathbf{a}$ and $\mathbf{b}$ in D, there exists a continuous curve C with $\mathbf{a}$ and $\mathbf{b}$ as endpoints that lies entirely in D. This is illustrated very simplistically in the figure below.

$D = A \cup B$ is connected

$D = A \cup B$ is not connected

We'll return to this idea in more detail later in the course. The important point for now is that when we work with an connected and open set D, we may talk about line integrals of vector fields over curves C that lie entirely in the set D.

Path-independence and the Fundamental Theorems of Calculus for Line Integrals

(Relevant section from AMATH 231 Course Notes: 2.3)

We begin with a couple of simple examples of line integrals of vector-valued functions, which will motivate the discussion.

Examples:

1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F} = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k}$ along the curve $\mathbf{g}(t) = (\cos t, \sin t, t)$, with $0 \le t \le 2\pi$.

   This parametrization produces a helical curve that starts at $(1, 0, 0)$ and ends at $(1, 0, 2\pi)$.

   Step 1: Evaluate the velocity vector: $\mathbf{g}'(t) = (-\sin t, \cos t, 1)$.

Step 2: Evaluate $\mathbf{F}$ at points on the curve:

$$\mathbf{F}(\mathbf{g}(t)) = (2x(t), 4y(t), z(t)) = (2\cos t, 4\sin t, t) \tag{25}$$

Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (2\cos t, 4\sin t, t) \cdot (-\sin t, \cos t, 1) = 2\cos t \sin t + t \tag{26}$$

Now evaluate the line integral:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} (2\cos t \sin t + t)\, dt \qquad (27) \\ &= \left[\sin^2 t + \frac{1}{2}t^2\right]_0^{2\pi} \\ &= 2\pi^2. \end{aligned}$$

2. Now evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F}$ is the vector field used in Example 1, but the curve is now the straight line from $(1, 0, 0)$ to $(1, 0, 2\pi)$. Since the value of the line integral is independent of the parametrization used, we'll use the simplest one, $\mathbf{g}(t) = (1, 0, t)$, with $0 \le t \le 2\pi$.

   Step 1: Evaluate the velocity vector: $\mathbf{g}'(t) = (0, 0, 1)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Here,

$$\mathbf{F}(\mathbf{g}(t)) = (2x(t), 4y(t), z(t)) = (2, 0, t). \tag{28}$$

   Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (2, 0, t) \cdot (0, 0, 1) = t \tag{29}$$

   Now evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} t\, dt = 2\pi^2. \tag{30}$$

Note that the results of Examples 1 and 2 are identical. This could be a coincidence but if you tried other paths with the same endpoints, you would obtain $2\pi^2$. We'll show very shortly that for vector field $\mathbf{F} = (x, 2y, 4z)$, the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{x} = 2\pi^2 \tag{31}$$

for *any* (piecewise C^1) path that starts at $(1,0,0)$ and ends at $(1,0,2\pi)$. In other words, **the line integral is independent of path** or simply **path-independent**.

The reason for this path-independence is the fact that the vector field $\mathbf{F}$ examined above is a **gradient field**, i.e., there exists a scalar function $f(x,y,z)$ such that $\mathbf{F} = \vec{\nabla} f$. (Recall that physicists prefer to think of a **conservative field**, i.e., a vector field $\mathbf{F} = -\vec{\nabla} V$ for some scalar function $V(x,y,z)$.) In this case,

$$\mathbf{F} = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k} = \vec{\nabla} f, \tag{32}$$

where

$$f(x,y,z) = x^2 + 2y^2 + \frac{1}{2}z^2 + C, \tag{33}$$

for any constant $C \in \mathbf{R}$.

Here is our claim, which is Theorem 2.2 in the AMATH 231 Course Notes, p. 45, the so-called **Second Fundamental Theorem for Line Integrals**:

Theorem: Let $\mathbf{F} : D \to \mathbf{R}^n$ be a continuous vector field on a connected and open set $D \subset \mathbf{R}^n$, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points in D. Furthermore, assume that $\mathbf{F}$ is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$, where $f : D \to \mathbf{R}$ is a C^1 scalar field. Now let C be **any** curve in D which joins $\mathbf{x}_1$ and $\mathbf{x}_2$ (in other words, the endpoints of C are $\mathbf{x}_1$ and $\mathbf{x}_2$) and let the orientation of the integration over C be from $\mathbf{x}_1$ (start) to $\mathbf{x}_2$ (finish). Then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \vec{\nabla} f \cdot d\mathbf{x} = f(\mathbf{x}_2) - f(\mathbf{x}_1) \qquad (f(\text{finish}) - f(\text{start}))\,. \tag{34}$$

Note that the line integral depends only on the endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ and not on the path C taken.

We'll often state this result as follows:

$$\int_{C_{AB}} \mathbf{F} \cdot d\mathbf{x} = \int_{C_{AB}} \vec{\nabla} f \cdot d\mathbf{x} = \text{“}f(B) - f(A)\text{”}, \tag{35}$$

where C_{AB} denotes a curve that starts at A and ends at B, and $f(A)$ and $f(B)$ denote the values of f at these respective endpoints.

Here, we can also comment that the above Theorem also implies the following relationship,

$$\int_{C_{BA}} \vec{\nabla} f \cdot d\mathbf{x} = f(A) - f(B) = -\left[f(B) - f(A)\right] = -\int_{C_{AB}} \vec{\nabla} f \cdot d\mathbf{x}\,, \tag{36}$$

where C_{BA} is any curve that starts at B and ends at A.

Note: In these lecture notes, we shall often refer to the Second Fundamental Theorem for Line Integrals in abbreviated form as "FTLI 2".

Why "Second Fundamental Theorem for Line Integrals"? Let's go back to the Second Fundamental Theorem of Calculus (FTC II) for functions of a single variable, which implies that

$$\int_a^b f'(x)\, dx = f(b) - f(a), \tag{37}$$

since $f(x)$ is an antiderivative of $f'(x)$. Comparing (37) and (34) it appears that the gradient of f, $\vec{\nabla} f$, is the "natural derivative" of a function f of several variables. (Of course, for the single variable case, it **is** the derivative of f: $\vec{\nabla} f = f'(x)\mathbf{i}$.)

Proof of the above Theorem: Let C be given by the parametrization

$$\mathbf{x}(t) = \mathbf{g}(t), \quad t_1 \leq t \leq t_2\,, \tag{38}$$

so that

$$\mathbf{x}_1 = \mathbf{g}(t_1), \quad \mathbf{x}_2 = \mathbf{g}(t_2)\,. \tag{39}$$

(Note: We do **not** have to come up with a particular parametrization. The knowledge that such a parametrization exists is sufficient for the proof.)

Then

$$\begin{aligned}
\int_C \vec{\nabla} f \cdot d\mathbf{x} &= \int_{t_1}^{t_2} \vec{\nabla} f(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}[f(\mathbf{g}(t))]\, dt \qquad \text{(Chain Rule)} \\
&= f(\mathbf{g}(t_2)) - f(\mathbf{g}(t_1)) \qquad \text{(FTC II for integrals over } \mathbf{R}\text{)} \\
&= f(\mathbf{x}_2) - f(\mathbf{x}_1)
\end{aligned} \tag{40}$$

and the theorem is proved.

At this point, you are probably saying, "Whoa! Wait! How did you get from Line No. 1 to Line No. 2?" Here is the explanation:

$$\vec{\nabla} f(\mathbf{g}(t)) = \left(\frac{\partial f}{\partial x_1}(\mathbf{g}(t)), \frac{\partial f}{\partial x_2}(\mathbf{g}(t)), \cdots, \frac{\partial f}{\partial x_n}(\mathbf{g}(t)) \right) \in \mathbf{R}^n \,. \tag{41}$$

and

$$\mathbf{g}(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathbf{R}^n \,, \tag{42}$$

so that

$$\mathbf{g}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t)) \in \mathbf{R}^n \,. \tag{43}$$

Therefore

$$\begin{aligned} \vec{\nabla} f(\mathbf{g}(t)) \cdot \mathbf{g}'(t) &= \sum_{k=1} \frac{\partial f}{\partial x_k}(\mathbf{g}(t)) \cdot x_k'(t) \\ &= \frac{d}{dt} f(\mathbf{g}(t)) \,. \end{aligned} \tag{44}$$

Revisiting Examples 1 and 2 above: Let us now return to the vector field $\mathbf{F}$ used in Examples 1 and 2, and the knowledge that $\mathbf{F}$ is a gradient field, as shown in Eq. (33). From the Second Fundamental Theorem for Line Integrals, the line integral will simply be the difference of the function f evaluated at the endpoints:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{x} &= \int_C \vec{\nabla} f \cdot d\mathbf{x} \\ &= f(1,0,2\pi) - f(1,0,0) \\ &= \left[x^2 + 2y^2 + \frac{1}{2} z^2 \right]_{(1,0,0)}^{(1,0,2\pi)} \\ &= 2\pi^2. \end{aligned} \tag{45}$$

As such, the value of the line integral will not depend on the path that is taken from $(1,0,0)$ to $(1,0,2\pi)$.

How do we know if a vector field is gradient/conservative?

Obviously, the Second FTLI (FTLI 2) gives a nice way of computing line integrals involving vector fields, but only if the vector fields are gradient/conservative. In fact, the consequence of independence of path is of fundamental importance to physics (next lecture), even if we are not so concerned about explicitly computing actual values of line integrals.

There naturally arise two questions:

1. If one is presented with a line integral involving a vector field $\mathbf{F}$, how does one know if $\mathbf{F}$ is a gradient/conservative field?

2. If $\mathbf{F}$ is indeed a gradient/conservative field, how do we find f/V?

Actually, we answered these questions in an earlier lecture:

Answer to 1: The vector field is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$ if

$$\vec{\nabla} \times \mathbf{F} = \mathbf{0}. \tag{46}$$

(Actually, we haven't defined the "curl" operation yet, but the above equation yields the conditions on the components F_i of $\mathbf{F}$ that we derived earlier, e.g.,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \tag{47}$$

for a vector field $\mathbf{F} = (F_1, F_2)$ in $\mathbf{R}^2$.

Answer to 2: We obtain f or V from $\mathbf{F}$ by means of a systematic procedure of partial antidifferentiation. So far, we've considered only the two-dimensional case, i.e., fields in $\mathbf{R}^2$.

Line integrals of vector-valued functions (cont'd)

Some important consequences of conservative fields in Physics

Recall that if a vector field $\mathbf{F}$ is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$, then it is also a conservative field, $\mathbf{F} = -\vec{\nabla} V$, with $V = -f$. (Actually, $V = -f + C$, where C is a constant.) Suppose that a conservative force acts upon a mass m while it is moving from point A to point B along a curve C_{AB} in $\mathbf{R}^3$. Then the total work done by the force is given by

$$\begin{aligned} W &= \int_{C_{AB}} \mathbf{F} \cdot d\mathbf{x} \qquad (48) \\ &= -\int_{C_{AB}} \vec{\nabla} V \cdot d\mathbf{x} \qquad (\mathbf{F} = -\vec{\nabla} V) \\ &= -[V(B) - V(A)] \qquad \text{(by FTLI 2)} \\ &= V(A) - V(B) \\ &= -\Delta V. \end{aligned}$$

Here, $V(A)$ and $V(B)$ denote the potential energies of the mass at A and B, respectively, and ΔV denotes the change in the potential energy from A to B. Note that we did not have to know the curve C_{AB} along which the mass travelled. The above result would have been valid for *any* curve that started at A and ended at B.

Now since $\mathbf{F}$ was assumed to be conservative, the total mechanical energy of the mass is conserved, i.e.,

$$K(A) + V(A) = K(B) + V(B), \qquad (49)$$

where $K(A)$ and $K(B)$ denote the kinetic energies of the mass at points A and B, respectively. Rearranging this equation, we have

$$V(A) - V(B) = K(B) - K(A), \qquad (50)$$

so that, from (48),

$$W = K(B) - K(A). \qquad (51)$$

The total work done by $\mathbf{F}$ is equal to the change in kinetic energy of the mass. This result was derived *for general forces, conservative and nonconservative alike,* in a previous lecture.

In summary, we have the following result for the total work done by the force $\mathbf{F}$ when the mass m

moves from A to B:

$$W = \Delta K = -\Delta V. \tag{52}$$

This makes sense: the total mechanical energy E of the mass remains constant. Any increase/decrease in its kinetic energy K must be accompanied by a decrease/increase in its potential energy V.

Here is another important consequence which is known to you from earlier courses in Physics. With reference to the previous discussion, in particular, Eq. (48), suppose that under the influence of the conservative force $\mathbf{F}$, the mass m moves from point A and, at some point in the future, returns to point A. In this case, $B = A$ so that the total work done by $\mathbf{F}$ is

$$W = V(A) - V(A) = 0\,. \tag{53}$$

In other words, **no net work is done by the force**.

An important note: At this point, we must be careful to recall the assumptions that were made about the conservative field $\mathbf{F}$ in order that the Second Fundamental Theorem for Line Integrals (FTLI 2) could be used to arrive at this conclusion: The scalar field f – in this case the potential field V – was assumed to be C^1, which implies that $\mathbf{F} = \vec{\nabla} f$ is **continuous** over the region $D \subset \mathbf{R^n}$.

This is important. In the case that the vector field $\mathbf{F}$ has **singularities**, i.e., points at which it is not continuous, then the result that "$W = 0$" is not necessarily true. We shall return to this topic very shortly.

We have already encountered a number of examples of physical forces that are conservative. For example, in an earlier lecture we discussed a two-dimensional mass-spring system, where the force exerted on a mass m was given by

$$\mathbf{F}(x, y) = -k_1 x\mathbf{i} - k_2 y\mathbf{j}, \tag{54}$$

where (x, y) denotes the position of the mass relative to the equilibrium point. This force is a conservative, and its associated potential energy function is

$$V(x, y) = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + C, \tag{55}$$

where C is a constant. Suppose that the mass were observed to be at position A with coordinates (a, b) at one time and at B (c, d) at another time. Then the net work done by the force in the net movement of the mass from A to B is, from Eq. (48),

$$W = V(A) - V(B) = \frac{1}{2}k_1(a^2 - c^2) + \frac{1}{2}k_2(b^2 - d^2). \tag{56}$$

If $W > 0$, then this amount is actually the work done *by* the force. If $W < 0$, then $|W|$ is the amount of work that has to be done *against* the force to move the mass from A to B.

As you well know by now, another class of forces that are conservative are those in $\mathbf{R}^3$ that have the form

$$\mathbf{F}(\mathbf{r}) = \frac{K}{r^3}\mathbf{r}, \tag{57}$$

where $r = \| \mathbf{r} \|$. Recalling that

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r}, \tag{58}$$

we have that $\mathbf{F} = -\vec{\nabla}V$, where

$$V(\mathbf{r}) = \frac{K}{r^3}\mathbf{r}. \tag{59}$$

1. $\mathbf{F}(\mathbf{r}) = \dfrac{Qq}{4\pi\epsilon_0 r^3}\mathbf{r}$, the electrostatic force on a charge q at $\mathbf{r}$ due to a charge Q at the origin $\mathbf{0}$. Here, $K = \dfrac{Qq}{4\pi\epsilon_0}$ so that the potential energy function is $V(\mathbf{r}) = \dfrac{Qq}{4\pi\epsilon_0 r}$.

2. $\mathbf{F}(\mathbf{r}) = -\dfrac{GMm}{r^3}\mathbf{r}$, the gravitational force on a mass m at $\mathbf{r}$ due to a mass M at the origin $\mathbf{0}$. Here, $K = -GMm$ so that the potential energy function is $V(\mathbf{r}) = -\dfrac{GMm}{r}$.

It's worth pointing out that earlier in this course, we "discovered" that these forces were conservative from the gradient relation in (58). We were spared the work of trying to find the potential functions associated with these forces, i.e., by first checking if the forces were conservative (using the curl test) and then integrating backwards to find the potential functions.

Here's an application that you've no doubt seen in first-year Physics: Suppose that a satellite moves from point A in space to point B along a curve C_{AB} under the influence of the earth's gravity. What is the work done by gravity?

We assume (and correctly so, as we'll prove later) that we can treat the earth as a point mass M that defines the origin O of our fixed coordinate system. Using the results from a couple of paragraphs above, the answer is simply

$$\begin{aligned} W = V(A) - V(B) &= -\frac{GMm}{r_A} + \frac{GMm}{r_B} \\ &= GMm\left[-\frac{1}{r_A} + \frac{1}{r_B}\right], \end{aligned} \tag{60}$$

where $r_A = |\overline{OA}|$ and $r_B = |\overline{OB}|$ are the radial distances between the earth (origin O) and the satellite at points A and B.

Some notes:

1. If $r_A = r_B$, then $W = 0$.

2. If $r_A > r_B$ (i.e., the satellite has moved inward), then $W > 0$. The work done by gravity is positive. This implies that $V(A) > V(B)$, i.e., there has been a *decrease* in potential energy. This, in turn, implies that there has been an *increase* in kinetic energy, i.e., $K(B) > K(A)$.

3. If $r_A < r_B$ (i.e., the satellite has moved outward), then $W < 0$. The work done by gravity is negative. In this case $V(A) < V(B)$, i.e., there has been an *increase* in potential energy which, in turn, implies a *decrease* in kinetic energy, i.e., $K(B) < K(A)$.

It is certainly possible that all of these situations can be encountered *during a single orbit* of the satellite around the earth. For example,

1. If the orbit is perfectly circular, then $r_A = r_B = r$, a constant, during the entire orbit, which means that no work is ever done.

2. If the orbit is elliptical, then pick two points on the orbit such that $r_A > r_B$. In travelling from A to B, work has been done *by* gravity (decrease in potential energy). In returning to A from B, an equal amount of work has been done *against* gravity (implying an equal increase in potential energy). The motion along an elliptical orbit involves a constant interchange between potential and kinetic energy.

Lecture 12

First Fundamental Theorem for Line Integrals (FTLI 1)

Recall the First Fundamental Theorem of Calculus (or FTC I): If a function f is continuous on [a,b], then the function g defined on $[a, b]$ as follows,

$$g(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b\,, \tag{61}$$

is an antiderivative of f, i.e.

$$g'(x) = f(x)\,. \tag{62}$$

In particular, g is the unique antiderivative of f for which

$$g(a) = 0\,. \tag{63}$$

As in the case of FLTI 2, there is a corresponding First Fundamental Theorem for Line Integrals, or "FTLI 1". The natural question is: What is the line integral analogue of Eq. (61) in which $f : \mathbf{R} \to \mathbf{R}$ is replaced by a vector field $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ and the points a and x in $\mathbf{R}$ are replaced by suitable points in $\mathbf{R}^n$?

The answer is that the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ must be path-independent. In this case, we define a scalar field $f : \mathbf{R}^n \to \mathbf{R}$ as follows,

$$f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{y}\,, \tag{64}$$

where $\mathbf{y}$ is a dummy integration variable in $\mathbf{R}^n$. Note that we do not have to specify a curve C over which the above integration is performed since the line integral is assumed to be independent of the path from $\mathbf{x}_0$ to $\mathbf{x}$. In this case, subject to some additional assumptions,

$$\vec{\nabla} f = \mathbf{F}\,. \tag{65}$$

In other words, $\mathbf{F}$ is a gradient field and f is viewed as a kind of antiderivative of $\mathbf{F}$.

We now state and prove the FTLI 1, which is Theorem 2.1 in the AMATH Course Notes, p. 43.

Theorem Let $\mathbf{F} : D \to \mathbf{R}^n$ be a continuous vector field on a connected and open subset $D \subset \mathbf{R}^n$. Furthermore assume that all line integrals over $\mathbf{F}$ are path independent in D. Now define the scalar-valued function $f : D \to \mathbf{R}^n$ as follows,

$$f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{y}\,, \tag{66}$$

for all $\mathbf{x} \in D$, where $\mathbf{x}_0$ is a specified point in D. Then

$$\vec{\nabla} f(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \quad \text{ for all } \ \mathbf{x} \in D\,. \tag{67}$$

Note: In these notes, we use f to denote the scalar field. In the AM231 Course Notes, ϕ is used to denote the scalar field.

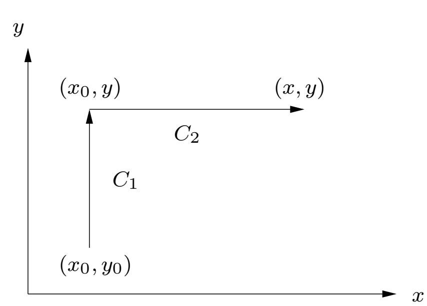
Proof: (We follow the proof presented in the AMATH Course Notes quite closely.) We consider the case $\mathbf{R}^2$ for simplicity. An extension of the proof to the general case $\mathbf{R}^n$ is possible, using the same basic ideas.

We must prove that for f defined in (66), Eq. (67) holds componentwise, i.e.,

$$\frac{\partial f}{\partial x} = F_1\,, \qquad \frac{\partial f}{\partial y} = F_2\,, \tag{68}$$

where F_1 and F_2 are the components of $\mathbf{F}$, i.e., $\mathbf{F} = (F_1, F_2)$.

Since line integrals over $\mathbf{F}$ in D are assumed to be path-independent, we may employ a special curve C that runs from $\mathbf{x}_0 = (x_0, y_0)$ to $\mathbf{x} = (x, y)$, both in D. The curve $C = C_1 \cup C_2$, composed of two segments, is shown in the figure below.



Important note: The initial point $\mathbf{x}_0$ can be any point in D. The fact that we have placed it below and to the left of the endpoint $\mathbf{x}$ – for purposes of convenience with regard to parametrization of curves – does **not** represent a loss of generality. Since all line integrals over $\mathbf{F}$ are assumed to be path independent in D, we could start at any point $\mathbf{x}_0 \in D$ and consider a path that eventually passes through a point $\mathbf{x}_1$ just below and to the left of endpoint $\mathbf{x}$. By additivity, the net line integral $f(\mathbf{x})$ would be the sum of the line integral of $\mathbf{F}$ from $\mathbf{x}_0$ to $\mathbf{x}_1$, a constant, and the line integral from $\mathbf{x}_1$ to $\mathbf{x}$. When we differentiate $f(\mathbf{x})$ partially with respect to x and y, the the constant term will disappear.

The following simple parametrizations for the segments comprising C will be employed:

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{g_1}(t) = (x_0, t), \quad y_0 \le t \le y\,, \qquad \text{Curve } C_1\,, \\
\mathbf{x}(t) &= \mathbf{g_2}(t) = (t, y), \quad x_0 \le t \le x\,, \qquad \text{Curve } C_2\,.
\end{aligned} \tag{69}$$

From the additivity property of the line integral,

$$\begin{aligned}
f(x,y) &= \int_C \mathbf{F} \cdot d\mathbf{y} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{y} + \int_{C_2} \mathbf{F} \cdot d\mathbf{y} \\
&= \int_{(x_0,y_0)}^{(x_0,y)} \mathbf{F} \cdot d\mathbf{y} + \int_{(x_0,y)}^{(x,y)} \mathbf{F} \cdot d\mathbf{y} \\
&= LI_1 + LI_2\,.
\end{aligned} \tag{70}$$

From the definition of the line integral, the component line integrals LI_1 and LI_2 over C_1 and C_2, respectively, may be written as follows,

$$LI_1 = \int_{y_0}^{y} \mathbf{F}(\mathbf{g}_1(t)) \cdot \mathbf{g}_1'(t)\, dt \tag{71}$$

and

$$LI_2 = \int_{x_0}^{x} \mathbf{F}(\mathbf{g}_2(t)) \cdot \mathbf{g}_2'(t)\, dt \tag{72}$$

We now compute the integrands in each component line integral, first computing the velocity vectors associated with each parametrization,

$$\mathbf{g}_1'(t) = (0,1), \qquad \mathbf{g}_2'(t) = (1,0)\,. \tag{73}$$

(These results should be clear from the figure above.) The integrand in LI_1 then becomes

$$(F_1(x(t),y(t)), F_2(x(t),y(t)) \cdot (0,1) = F_2(x(t),y(t)) = F_2(x_0,t)\,, \tag{74}$$

and the integrand in LI_2 becomes

$$(F_1(x(t),y(t)), F_2(x(t),y(t))) \cdot (1,0) = F_1(x(t),y(t)) = F_1(t,y)\,. \tag{75}$$

Substitution into (71) and (72) yields

$$LI_1 = \int_{y_0}^{y} F_2(x_0, t)\, dt\,, \quad LI_2 = \int_{x_0}^{x} F_1(t, y)\, dt\,. \tag{76}$$

From Eq. (70), we have

$$f(x, y) = \int_{y_0}^{y} F_2(x_0, t)\, dt \ + \ \int_{x_0}^{x} F_1(t, y)\, dt\,. \tag{77}$$

Note that the first definite integral in Eq. (77) is independent of x. We take the partial derivatives of both sides with respect to x, treating, of course, y as a constant, to obtain

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\int_{y_0}^{y} F_2(x_0, t)\, dt\right] + \frac{\partial}{\partial x}\left[\int_{x_0}^{x} F_1(t, y)\, dt\right]\,. \tag{78}$$

The first term on the RHS is zero. Recalling that y is kept constant during partial differentiation with respect to x, the second term may be evaluated by means of the first Fundamental Theorem of Calculus for Riemann integrals to yield the result,

$$\frac{\partial f}{\partial x} = F_1(x, y)\,. \tag{79}$$

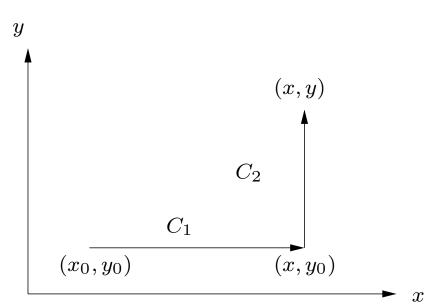
We have accomplished one-half of our goal in Eq. (68).

Unfortunately, if we try to compute the other partial derivative from (77),

$$\frac{\partial f}{\partial y} = F_2(x_0, y) + \int_{x_0}^{x} \frac{\partial F_1}{\partial y}(t, y)\, dt\,, \tag{80}$$

we can proceed no further, i.e., we cannot show that the RHS is equal to $F_2(x, y)$. In fact, it is not even guaranteed that the integral in this equation exists. The only assumption on the vector field $\mathbf{F}$ is that it is continuous. As such, there is no guarantee that the integrand $\dfrac{\partial F_1}{\partial y}$ is a continuous function, or that it even exists!

Fortunately, the other relation in Eq. (68) may be derived if we employ a different integration path – the one sketched in the figure below.



By means of a procedure quite analogous to the one used above – which is left as an exercise for the reader – we arrive at the desired result, namely,

$$\frac{\partial f}{\partial y} = F_2(x, y)\,, \tag{81}$$

and the proof is complete.

A note regarding Eqs. (80) and (81) in the above proof

If we make the additional assumption that the vector field $\mathbf{F}$ in the above Theorem is C^1, i.e., has continuous partial derivatives, then the integral in Eq. (80) exists. From Eqs. (80) and (81), it then follows that

$$F_2(x, y) = F_2(x_0, y) + \int_{x_0}^{x} \frac{\partial F_1}{\partial y}(t, y)\, dt\,, \tag{82}$$

which may seem to be a rather strange result.

Note that the first term on the RHS of Eq. (82) is independent of x. Now take the partial derivatives of both sides of this equation with respect to x to obtain

$$\begin{aligned} \frac{\partial F_2}{\partial x}(x, y) &= 0 + \frac{\partial}{\partial x}\left[\int_{x_0}^{x} \frac{\partial F_1}{\partial y}(t, y)\, dt\right] \\ &= \frac{\partial F_1}{\partial y}(x, y)\,. \end{aligned} \tag{83}$$

Recall that we derived this condition – assuming that the partial derivatives are continuous functions - for the vector field $\mathbf{F}$ to be a gradient field, i.e., that $\mathbf{F} = \vec{\nabla} f$ for a scalar field f.

Unfortunately, we could not use this result in the proof, since no assumption was made on the existence of partial derivatives of the F_i.

Another side note

The above note leads to the following point: Earlier in this course, we derived some necessary conditions to be satisfied by the components of a vector field $\mathbf{F}$ in order for it to be a gradient or conservative field. For example, for a vector field in $\mathbf{R^2}$, the condition is

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}\,. \tag{84}$$

This condition is possible only with the additional assumption that $\mathbf{F}$ is a C^1 vector field, i.e., its partial derivatives are continuous functions over a region $D \in \mathbf{R^n}$. A natural question (which a couple of students asked at the end of the lecture) is: "What happens if our vector field $\mathbf{F}$ is not C^1? How can we determine whether or not it is gradient/conservative?"

Without trying to sound evasive, the answer depends on what "not C^1" means. If the vector field is C^1 over a region D with the exception of some "small" sets, e.g., points, curves, then the condition involving its partial derivatives may still apply over all other points of D. This is, in fact, the case when we consider gravitational or electrostatic fields involving point masses or charges – we simply avoid the points at which these points masses/charges are located. This is usually quite sufficient in applications.

That being said, a mathematician (or perhaps even a mathematical physicist) may wish, for some reason, to consider rather "pathological" vector fields, for example, those which are not differentiable at any point. This would require other methods of analysis which are quite far from those of standard, "classical" calculus.

A final note regarding the proof of FTLI 1

As mentioned in class, the proof of the FTLI shown above is quite "nuts and bolts", in which you consider a specific path and then "grind out" the needed result. It relies on a specific set of constructions which may actually seem quite contrived. This is an example of a "constructive proof." Even so, it is a valid proof. We'll find that constructive proofs will be used to derive a number of important results in vector calculus.

If you go back to the proof, in particular, the use of the first set of curves C_1 and C_2 used to establish Eq. (79), you'll see that the choice of curves was quite clever. The curve C_1 does not depend on a variable x. The curve C_2 approaches the point (x, y) by keeping y fixed and varying only the first coordinate. This makes it possible to compute the partial derivative $\dfrac{\partial f}{\partial x}$.

The second set of curves then reverse this strategy by approaching the point (x, y) along a line over which the first coordinate is kept fixed. This makes it possible to compute the partial derivative $\dfrac{\partial f}{\partial y}$.

The reader may now get an idea of how to treat the higher dimensional case, e.g., $\mathbf{R^3}$.

Application of FTLI 1 to Physics

Recall that in Physics, it is more convenient to consider a vector field $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$, often a force field, as a **conservative** field instead of a gradient field. Let us recall that a C^1 vector field $\mathbf{F}$ is conservative if the partial derivatives of its components satisfy a set of relations. For example, in $\mathbf{R}^2$, where $\mathbf{F} = (F_1, F_2)$, these relations are

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \,. \tag{85}$$

In this case, there exists a scalar field $V : \mathbf{R}^n \to \mathbf{R}$, commonly known as the **potential** associated with $\mathbf{F}$, such that

$$\mathbf{F} = -\vec{\nabla} V \quad (= \vec{\nabla} f) \,. \tag{86}$$

(Once again, we simply replace f of the previous section with $-V$.)

We have already discussed the consequences of the Second Fundamental Theorem for Line Integrals for conservative forces in a previous section. Here, we examine the implications of FTLI 1. If $\mathbf{F}$ is conservative, then we may define the associated potential function $V(\mathbf{x})$ as follows,

$$f(\mathbf{x}) = -V(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{y} \,, \tag{87}$$

where $\mathbf{x}_0$ is a suitable reference point. This then leads to

$$V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{y} \,. \tag{88}$$

Once again, the value of the potential function V at $\mathbf{x}$ may be interpreted as **the work done against the force F in moving a mass from the reference point $\mathbf{x}_0$ to point $\mathbf{x}$**. No curve C need be specified in the above line integral because the line integral is path-independent. Note also that

$$V(\mathbf{x}_0) = 0 \,. \tag{89}$$

From Eq. (88) and the FTLI 1, it follows that

$$\vec{\nabla} V = -\mathbf{F} \,, \tag{90}$$

from which Eq. (86) follows.

Eq. (88) is the n-dimensional extension of the one-dimensional result that we discussed in an earlier lecture: If a force $\mathbf{F} = f(x)\mathbf{i}$ is acting on a mass that can move only in one dimension (represented by the x-axis), then the potential energy associated with this force is

$$V(x) = -\int_a^x f(s)\, ds \,, \tag{91}$$

where a is a suitable reference point. Furthermore, from the First Fundamental Theorem of Calculus for Riemann integrals,

$$V'(x) = -f(x) \quad \Longrightarrow \quad f(x) = -V'(x)\,. \tag{92}$$

Line integrals of vector fields over closed curves

Let us recall the **Second Fundamental Theorem for Line Integrals (FTLI 2)**, proven in a previous section:

Theorem: Let $\mathbf{F} : D \to \mathbf{R}^n$ be a continuous vector field on a connected and open set $D \subset \mathbf{R}^n$, and let $\mathbf{a}$ and $\mathbf{b}$ be any two points in D. Furthermore, assume that $\mathbf{F}$ is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$, where $f : D \to \mathbf{R}$ is a C^1 scalar field. Now let C be **any** curve in D which joins $\mathbf{a}$ and $\mathbf{b}$ (in other words, the endpoints of C are $\mathbf{a}$ and $\mathbf{b}$) and let the orientation of the integration over C be from $\mathbf{a}$ (start) to $\mathbf{b}$ (finish). Then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \vec{\nabla} f \cdot d\mathbf{x} = f(\mathbf{b}) - f(\mathbf{a}) \qquad (f(\text{finish}) - f(\text{start}))\,. \tag{93}$$

As mentioned in the previous section, the line integral depends only on the endpoints $\mathbf{a}$ and $\mathbf{b}$ and not on the path C taken.

We now ask the question, what happens if we start from $\mathbf{a}$, go away for a while (e.g., sixth floor of MC, Student Life Centre, etc.) and then return to $\mathbf{a}$. This, of course, implies that $\mathbf{b} = \mathbf{a}$ and the above result becomes

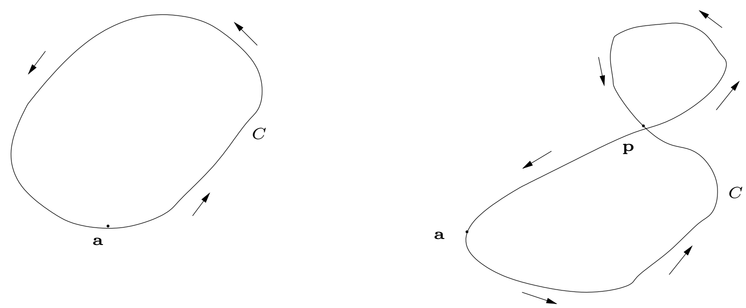
$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \vec{\nabla} f \cdot d\mathbf{x} = f(\mathbf{a}) - f(\mathbf{a}) = 0\,. \tag{94}$$

This will be the case if the conditions of the theorem are satisfied, i.e., the vector field $\mathbf{F}$ is continuous and a gradient field over the region that you have travelled.

If $\mathbf{F}$ is a conservative force, e.g., gravity, in which case $f = -V$, where V is the potential energy function, then the above leads to the important conclusion that **no net work was done**, either by the force or against the force.

Closed curves and simple closed curves

In these situations, where we finish at the same point $\mathbf{a}$ from which we started, the line integral is being performed over a **closed curve**, C, as sketched below. If the curve $C \in \mathbf{R}^n$ does not intersect itself, as on the left, then C is called a **simple curve**. If it intersects itself, then it it is **nonsimple**, as sketched on the right.



Left: Simple closed curve. **Right:** Nonsimple closed curve.

It is possible that during the course of a line integration over a curve C, a point $\mathbf{p} \in \mathbf{R}^n$, or even several points $\mathbf{p}_k$, are revisited before returning to $\mathbf{a}$. Such points are intersection points of the curve. For much of this course, however, we shall be concerned mostly with line integrals of vector fields over simple closed curves (in $\mathbf{R}^n$).

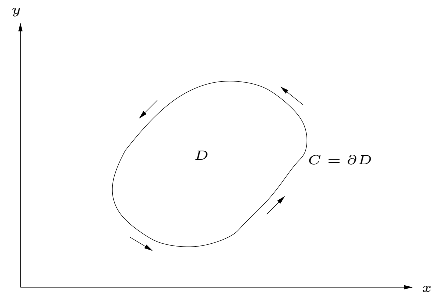
The usual notation for a line integral over a simple closed curve C is

$$\oint_C \mathbf{F} \cdot d\mathbf{x} \,. \tag{95}$$

(That being said, this notation is not employed in the AMATH 231 Course Notes, but it will be employed in these lecture notes.)

It is also important to specify the orientation of the closed curve C, i.e., the direction of the path of integration taken. In the plane $\mathbf{R}^2$, the standard approach is to travel over a simple closed curve C in a **counterclockwise direction**, so that the interior region enclosed by C lies **to the left**, as sketched below.

In fact, the situation of a simple closed curve C in the plane is a rather special one, since the curve C divides the plane into two non-overlapping regions, namely, (i) the interior region D enclosed by C, and assumed not to include C, and (ii) the exterior region E lying outside C and not including it. In this convention – which is followed by the AMATH 231 Course Notes (Page 48, item (iii)) – the set D is an open set. The boundary of D, which is usually denoted as ∂D is, in fact, curve C.



Sometimes, the union of the two sets, i.e., D and its boundary $\partial D = C$ will be used in a discussion or theorem. In the course notes, this is simply written as the set $D \cup \partial D$. This set is also known as the **closure** of the open set D and is often written as

$$\bar{D} = \text{“closure}(D)\text{”} = D \cup \partial D\,. \tag{96}$$

In this case, the closure of the set D is accomplished by including its boundary points. (The one-dimensional analogue is closing the open interval (a, b) by including its two boundary points a and b to produce the closed interval $[a, b]$.)

Some final comments about line integrals over closed curves, with an eye to what lies ahead ...

Let's return to the idea of performing a line integration of what seems to be a gradient or conservative field $\mathbf{F}$ over a simple closed curve C. It may well be the case that the result in Eq. (105) is not correct, i.e., a nonzero result is achieved. This does **not** imply that that FTLI 1 stated earlier is incorrect. What may be happening is that the conditions of the theorem are not being satisfied. Many vector fields in Physics have **singularities**, that is, points at which the fields are perhaps not defined, or not continuous, or differentiable. For example, the electrostatic field $\mathbf{E}(\mathbf{r})$ produced by a point charge Q situated at the origin,

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r^3}\,\mathbf{r}, \tag{97}$$

where $\mathbf{r} = (x, y, z)$ and $r = \|\mathbf{r}\|$, is undefined at $(0, 0, 0)$, the location of the point charge Q. (Part of the problem is that the idea of a “point charge” is a mathematical idealization – in nature, there really is no such thing as a “point charge” where a nonzero amount of electrical charge is situated at a single point of zero volume. Even in the case of an electron, its charge is “smeared out.” Nevertheless, it is often convenient to work with such idealizations and still come up with correct answers.)

In such cases, a physical vector field $\mathbf{F}$ may actually be conservative except at its singular points. For this reason, the region D over which line integrals (and later, surface integrals) involving $\mathbf{F}$ are performed will have to be restricted in order to be able to guarantee that line integrals of $\mathbf{F}$ over all simple closed curves in D are zero.

Circulation of a vector field around a closed curve C in R^2

In this section, we consider the line integral of a planar vector field $\mathbf{F}$ around a simple closed curve C in $\mathbf{R}^2$, denoted as

$$\oint_C \mathbf{F} \cdot d\mathbf{x} \,. \tag{98}$$

The convention is that the integration along C is performed in the counterclockwise direction so that the region D enclosed by C lies always to the left of C as we move along the curve.

Assuming that we can parametrize the closed curve C as $\mathbf{x}(t) = \mathbf{g}(t)$, $a \le t \le b$, with $\mathbf{g}(a) = \mathbf{g}(b)$, the line integral is normally computed as follows,

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t))) \cdot \mathbf{g}'(t)\, dt \tag{99}$$

We'll perform a few practical computations shortly. At this time, however, let us make a few modifications to the above equation in order to discover some deeper meaning of this line integral:

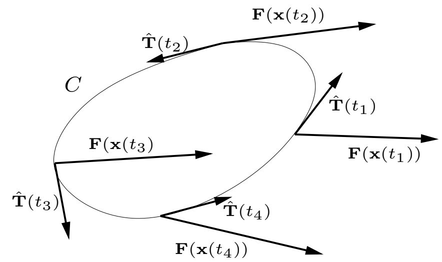
$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt \\ &= \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \frac{\mathbf{g}'(t)}{\parallel \mathbf{g}'(t) \parallel} \parallel \mathbf{g}'(t) \parallel dt \\ &= \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \hat{\mathbf{T}}(t)\, ds \\ &= \oint f ds, \end{aligned} \tag{100}$$

where the scalar-valued function,

$$f(\mathbf{g}(t)) = \mathbf{F}(\mathbf{g}(t)) \cdot \hat{\mathbf{T}}(t) \,, \tag{101}$$

is the projection of $\mathbf{F}$ in the direction of the unit tangent vector to the curve C at $\mathbf{r}(t)$, as sketched below.

Starting at any point P on the curve C, the orientation of the tangent vector $\hat{\mathbf{T}}$ will change as we travel along C. In one traversal of C, the net rotation of the tangent vector is 2π. This is quite

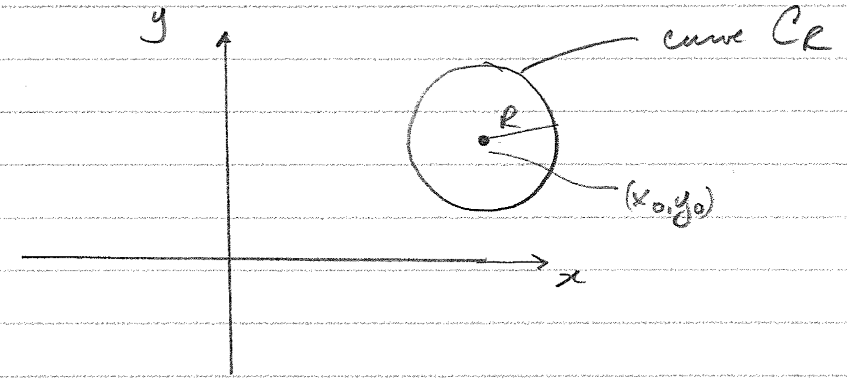


clear when C is a circle. The line integral in (100) sums up the projection of the vector field $\mathbf{F}(\mathbf{g}(t))$ onto the unit tangent vector $\hat{\mathbf{T}}(t)$ to the curve. As such, we say that the line integral in (100) is the *circulation of the vector field* $\mathbf{F}$ *around the closed curve* C.

If the vector field $\mathbf{F}$ is roughly parallel over the region D enclosed by curve C, then we expect the line integral to be small in value – in some regions of the curve, $\mathbf{F}$ points in the same direction as $\hat{\mathbf{T}}$ and in others, it points in the opposite direction. In other words, the vector field exhibits very little circulation.

Lecture 13 — Circulation Integrals In The Plane (cont'd)

Let's now compute a few circulation integrals. In the following examples, we'll integrate over a circle of radius R centered at a point (x_0, y_0) in the plane, ie



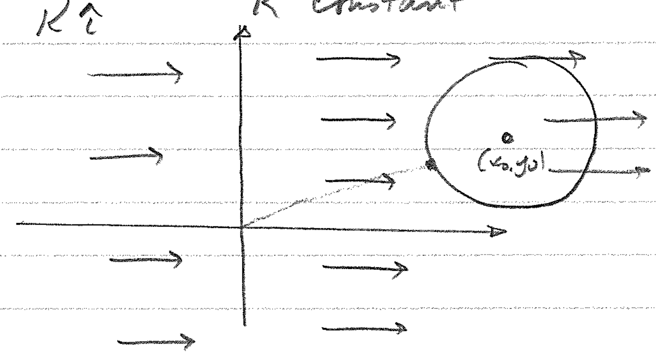
A parametrization for this curve is

$$\vec{g}(t) = (x_0 + R\cos t,\ y_0 + R\sin t) \qquad 0 \le t \le 2\pi$$

$$\Rightarrow \quad \vec{g}\,'(t) = (-R\sin t,\ R\cos t)$$

Example 1

$$\vec{F}(x,y) = K\hat{\imath} \qquad K \text{ constant} \qquad (x,y) \in \mathbb{R}$$



We suspect that the net circulation over any circle C_R will be zero. But let's do the math!

Evaluate $\vec{F}(\vec{g}(t))$ over the curve

$$\vec{F}(\vec{g}(t)) = \vec{F}(x(t), y(t)) = (K, 0)$$

Thus the integrand of the line integral will be

$$\vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = (K,0)\cdot(-R\sin t, R\cos t)$$

$$= -RK\sin t$$

Now compute the line integral

$$\oint_{C_R} \vec{F}\cdot d\vec{x} = \int_0^{2\pi} \vec{F}(\vec{g}(t))\cdot \vec{g}'(t)\,dt$$

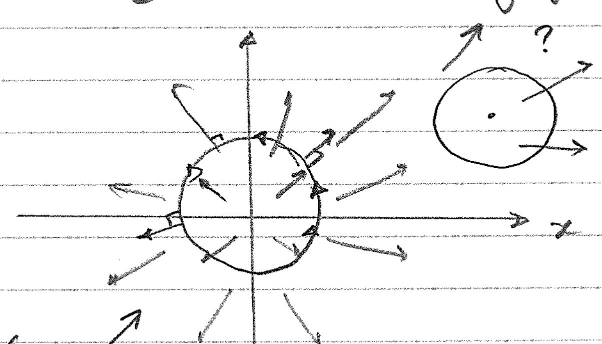
$$= \int_0^{2\pi} [-RK\sin t]\,dt$$

$$= 0,$$

as expected.

<u>Example 2</u> The vector field

$$\vec{F}(x,y) = Kx\,\hat{i} + Ky\,\hat{j} \qquad (x,y)\in\mathbb{R}^2$$



$\vec{F}$ is a radial field. Zero circulation?

In the special case $(x_0, y_0) = (0,0)$
the vector field is orthogonal to tangent vector $\vec{g}'(t)$ at all points on the curve. We expect circulation to be zero.

Compute $\vec{F}$ over curve C_R

$$\vec{F}(\vec{g}(t)) = \vec{F}(x(t), y(t))$$

$$= K(x_0 + R\cos t,\ y_0 + R\sin t)$$

$$\Rightarrow \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = K(x_0 + R\cos t,\ y_0 + R\sin t) \cdot (-R\sin t,\ R\cos t)$$

$$= -Kx_0 R\sin t - KR^2 \cos t \sin t$$
$$+ Ky_0 R\cos t + KR^2 \sin t \cos t$$

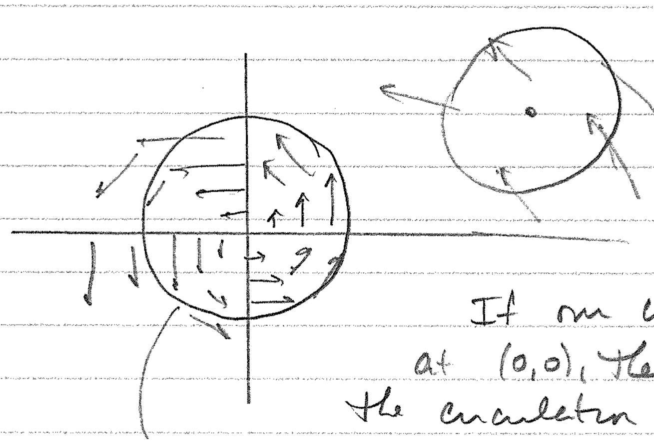
$$= -Kx_0 R\sin t + Ky_0 R\cos t$$

$$\Rightarrow \oint_{C_R} \vec{F} \cdot d\vec{x} = KR \int_0^{2\pi} [-x_0 \sin t + y_0 \cos t]\, dt = 0$$

zero circulation

Example 3: The rotating turntable vector field

$$\vec{F} = -Ky\hat{i} + Kx\hat{j} \qquad (x,y) \in \mathbb{R}$$



C_R

If our circle C_R is centered at $(0,0)$, then it appears that the circulation is quite high.
But what happens at (x_0, y_0) away from $(0,0)$?

Once again, evaluate $\vec{F}(g(t))$

$$\vec{F}(x(t), y(t)) = \left(-Ky(t), Kx(t)\right)$$

$$= K\left(-y_0 - R\sin t, x_0 + R\cos t\right)$$

$$\Rightarrow \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t) = K\left(-y_0 - R\sin t, x_0 + R\cos t\right) \cdot \left(-R\sin t, R\cos t\right)$$

$$= K\left\{y_0 R\sin t + R^2\sin^2 t - x_0 R\sin t + R^2\cos^2 t\right\}$$

$$= K\left\{y_0 R\sin t - x_0 R\sin t + R^2\right\}$$

$$\Rightarrow \oint_{C_R} \vec{F} \cdot d\vec{x} = K\int_0^{2\pi}\left[\cancelto{0}{y_0 R\sin t} - \cancelto{0}{x_0 R\sin t} + R^2\right]dt$$

$$= 2\pi K R^2$$

Net result: The circulation of $\vec{F} = -Ky\hat{i} + Kx\hat{j}$

around a circle of radius R and centered at (x_0, y_0)

is $\underline{\underline{2\pi K R^2}}$

Note that this result does <u>not</u> <u>depend</u> <u>upon</u> the <u>location of the circle.</u>

There's something suspicious to this result:

$$2\pi K R^2 = 2K\underbrace{(\pi R^2)}_{\text{area of circle.}}$$

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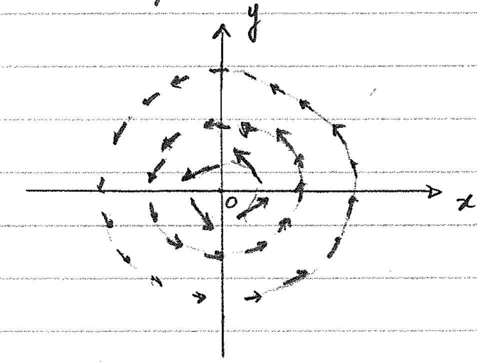
Example 4: The modified rotating vector field

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j} \qquad \underbrace{(x,y) \neq (0,0)}_{!!!}$$

(Recall from Problem Set No. 2:
The vector field $\frac{\mu_0 I}{2\pi}\vec{F}(x,y)$ is the magnetic field around a thin wire located on the z-axis and conducting a current $\vec{I} = I\hat{k}$.)



Also recall: $\|\vec{F}(x,y)\| = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{d}$ d- distance to $(0,0)$

Important point $\vec{F}(0,0)$ is undefined.

$\Rightarrow$ $(0,0)$ is a singular point.

Once again, we consider circle of radius R centered at (x_0, y_0)

$$\vec{x}(t) = \vec{g}(t) = (x_0 + R\cos t,\; y_0 + R\sin t) \qquad 0 \le t \le 2\pi$$

$$\Rightarrow \vec{g}\,'(t) = (-R\sin t,\; R\cos t)$$

Evaluate $\vec{F}(\vec{g}(t))$:

For simplicity: $x(t)^2 + y(t)^2 = (x_0 + R\cos t)^2 + (y_0 + R\sin t)^2$

$$= x_0^2 + y_0^2 + R^2 + 2R(x_0\cos t + y_0 \sin t)$$

$$=: D$$

$$\vec{F}(g(t)) = \vec{F}(x(t), y(t)) = \left(-\frac{y_0 + R\sin t}{D}, \frac{x_0 + R\cos t}{D} \right)$$

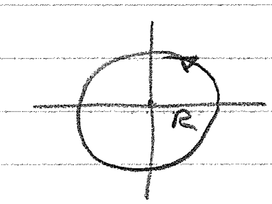
$$\Rightarrow \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = \left(-\frac{y_0 + R\sin t}{D}, \frac{x_0 + R\cos t}{D} \right) \cdot (-R\sin t, R\cos t)$$

$$= \frac{1}{D}\left[(y_0 + R\sin t) R \sin t + (x_0 + R\cos t) R\cos t \right]$$

$$= \frac{y_0 R \sin t + x_0 R \sin t + R^2}{D}$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{x} = \int_0^{2\pi} \frac{R(x_0\cos t + y_0 \sin t) + R^2}{x_0^2 + y_0^2 + R^2 + 2R(x_0 \cos t + y_0 \sin t)}\, dt \quad (*)$$

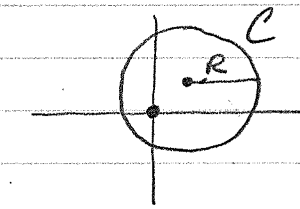
Special case: $x_0 = y_0 = 0$



Circle of radius R centered at $(0,0)$

$$\oint \vec{F} \cdot d\vec{x} = \int_0^{2\pi} 1 \, dt = 2\pi$$

After some calculations (details are presented later), we find:



Case 1: $x_0^2+y_0^2 < R^2$

In this case, circle C encloses the singular point $(0,0)$

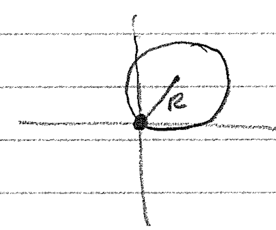
$$\boxed{\oint \vec{F}\cdot d\vec{x} = 2\pi} \quad (1)$$

Note that there is no dependence on R (this is a consequence of $\|\vec{F}\|$ decreasing at the "right" rate of a physical force)

Case 2: $x_0^2+y_0^2 = R^2$

In this case, circle C contains the singular point $(0,0)$

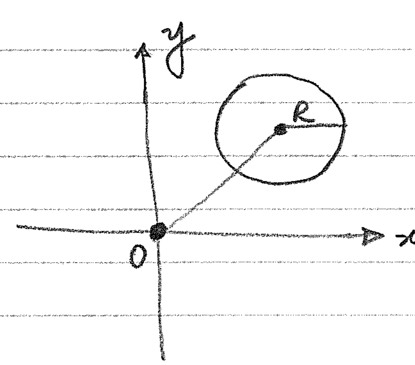
$$\boxed{\oint \vec{F}\cdot d\vec{x} \text{ is undefined.}} \quad (2)$$



Case 3: $x_0^2+y_0^2 > R^2$

Circle C does not enclose or contain the singular point $(0,0)$

$$\boxed{\oint \vec{F}\cdot d\vec{x} = 0} \quad (3)$$



Very shortly, we'll be able to explain these results, especially Case No. 3.

ASIDE: Details of computation of integral in Eq. (*), page 11.6

(You don't have to learn this!)

$$\oint \vec{E}\cdot d\vec{x} = \int_0^{2\pi} \frac{R(x_0\cos t + y_0 \sin t) + R^2}{x_0^2 + y_0^2 + R^2 + 2R(x_0 \cos t + y_0 \sin t)}\, dt$$

First of all, simplify $x_0 \cos t + y_0 \sin t = r\cos(t-\phi)$

where $r = \sqrt{x_0^2+y_0^2}$ $\qquad \tan\phi = \frac{y_0}{x_0}$

$$\left(x_0 \cos t + y_0 \sin t = \sqrt{x_0^2+y_0^2}\left[\underbrace{\frac{x_0}{\sqrt{x_0^2+y_0^2}}}_{\cos\phi} \cos t + \underbrace{\frac{y_0}{\sqrt{x_0^2+y_0^2}}}_{\sin\phi} \sin t\right]\right)$$

Since we're integrating over $0 \le t \le 2\pi$, and trig terms in numerator & denominator are identical, we can ignore ϕ.

Integral in (*) becomes

$$I = \int_0^{2\pi} \frac{Rr\cos t + R^2}{r^2 + R^2 + 2rR\cos t}\, dt$$

Must consider two cases.

Case A: $r < R$ $\qquad$ Divide num & denom by R^2

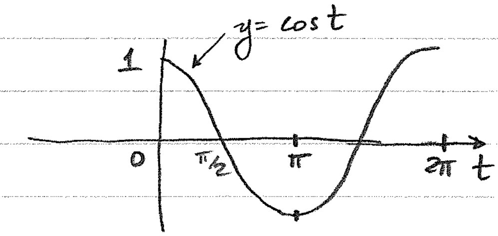
$$I = \int_0^{2\pi} \frac{a\cos t + 1}{1 + a^2 + 2a\cos t}\, dt \qquad a = \frac{r}{R} < 1$$

$$= a \int_0^{2\pi} \frac{\cos t}{1+a^2+2a\cos t}\, dt + \int_0^{2\pi} \frac{1}{1+a^2+2a\cos t}\, dt$$

$$I = 2a \int_0^{\pi} \frac{\cos t}{1+a^2+2a\cos t}\, dt + 2\int_0^{\pi} \frac{1}{1+a^2+2a\cos t}\, dt \quad (**)$$

From Table of Integrals, Series & Products (I.S. Gradshteyn & I.M. Ryzhik)

$$\int_0^{\pi} \frac{\cos nt}{1+\alpha^2-2\alpha\cos t}\, dt = \frac{\pi\alpha^n}{1-\alpha^2} \qquad \alpha^2 < 1$$



Set $\alpha = -a$. First integral in (**), $n=1$:

$$\int_0^{\pi} \frac{\cos t}{1+a^2+2a\cos t}\, dt = \frac{\pi(-a)}{1-a^2}$$

$$= -\frac{a\pi}{1-a^2}$$

(cos t < 0 for $\frac{\pi}{2} < t < \pi$ denominator is smaller $\Rightarrow$ integral is negative)

Second integral in (**), $n=0$:

$$\int_0^{\pi} \frac{1}{1+a^2+2a\cos t}\, dt = \frac{\pi}{1-a^2}$$

Net result

$$I = \frac{\pi}{1-a^2}\left(-2a^2+2\right) = 2\pi .$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{x} = 2\pi \quad \text{when} \quad r < R$$

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Case B: $r > R$ We can still use Eq. (**)

From Table of Integrals ... $a = \frac{r}{R}$

$$\int_0^{\pi} \frac{\cos nt}{1+\alpha^2 - 2\alpha \cos t}\, dt = \frac{\pi}{(\alpha^2-1)\alpha^n} \qquad a^2 > 1$$

Again set $\alpha = -a$.

First integral in (**), $n=1$:

$$\int_0^{\pi} \frac{\cos t}{1+a^2+2a\cos t}\, dt = \frac{\pi}{(a^2-1)(-a)} = -\frac{\pi}{a(a^2-1)}$$

Second integral in (**), $n=0$:

$$\int_0^{\pi} \frac{1}{1+a^2+2a\cos t}\, dt = \frac{\pi}{a^2-1}$$

Net result:

$$I = \frac{2\pi}{a^2-1}\left[-\frac{a}{a}+1\right] = 0$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{x} = 0 \quad \text{when} \quad r > R.$$

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To understand these results, we now move to

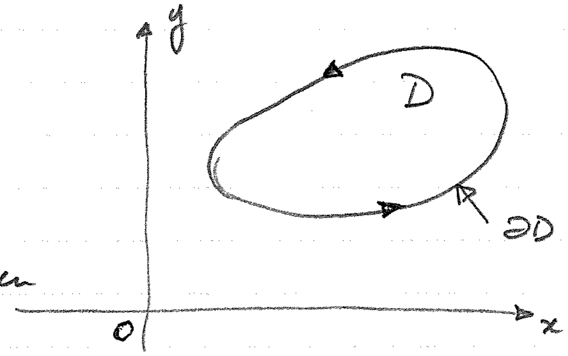
GREEN'S THEOREM (in the plane)

(AMATH 231 Course Notes, p. 49-53)

It concerns line integrals of "nice" (C^1) vector fields over "nice" (piecewise C^1) simple closed curves in the plane.

Green's Theorem

Let D be a bounded subset of $\mathbb{R}^2$ with boundary ∂D that is a piecewise simple closed curve oriented counter-clockwise



If $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$ is of class C^1 on $D \cup \partial D$ then

$$\underbrace{\int_{\partial D} \vec{F} \cdot d\vec{x}} = \underbrace{\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA}$$

line integral over boundary curve ∂D

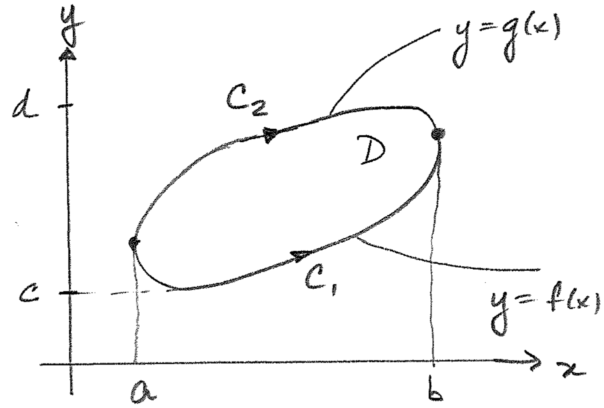
2D integration over region D enclosed by ∂D

circulation integral of F over curve $C = \partial D$

We shall consider the simplest case for region D - that it is described by inequalities of the form

$$h(y) \le x \le k(y) \qquad a \le y \le b$$

$$f(x) \le y \le g(x) \qquad a \le x \le b$$



$C = \partial D = C_1 \cup C_2$ where

C_1 : $y = f(x) \quad a \le x \le b$ or, in parametrized form

$$\vec{g}_1(x) = (x, f(x)) \qquad a \le x \le b$$

Note that we're going to try to use "x" as parameter.

C_2 : $y = g(x) \quad a \le x \le b$

$$\vec{g}_2(x) = (x, g(x)) \qquad a \le x \le b$$

Note orientation of C_2 - we'll have to correct for this

Then
$$\int_C \vec{F}\cdot d\vec{x} = \int_{C_1} \vec{F}\cdot d\vec{x} - \int_{C_2} \vec{F}\cdot d\vec{x}$$

$$= \underbrace{\int_a^b \vec{F}(\vec{g}_1(x))\cdot \vec{g}_1'(x)\,dx}_{①} - \underbrace{\int_a^b \vec{F}(\vec{g}_2(x))\cdot \vec{g}_2'(x)\,dx}_{②} \qquad (1)$$

①: $\vec{g}_1'(x) = (1, f'(x))$

②: $\vec{g}_2'(x) = (1, g'(x))$

Integrands:

①: $\vec{F}(\vec{g}_1(x)) \cdot \vec{g}_1'(x) = \left(F_1(x, f(x)), F_2(x, f(x))\right) \cdot (1, f'(x))$

$$= F_1(x, f(x)) + F_2(x, f(x)) f'(x)$$

②: $\vec{F}(\vec{g}_2(x)) \cdot \vec{g}_2(x) = \left(F_1(x, g(x)), F_2(x, g(x))\right) \cdot (1, g'(x))$

$$= F_1(x, g(x)) + F_2(x, g(x)) \cdot g'(x)$$

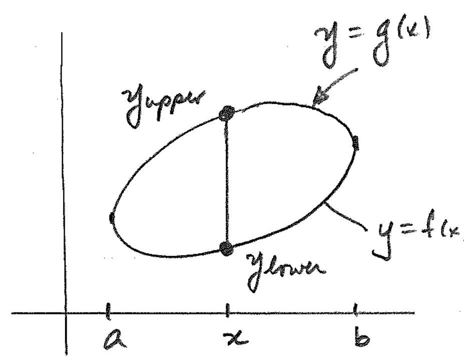
Substitute into (1) ...

$$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \left[F_1(x, f(x)) + F_2(x, f(x)) f'(x)\right] dx - \int_a^b \left[F_1(x, g(x)) + F_2(x, g(x)) g'(x)\right] dx$$

$$= \underbrace{\int_a^b \left[F_1(x, f(x)) - F_1(x, g(x))\right] dx}_{③} + \underbrace{\int_a^b \left[F_2(x, f(x)) f'(x) - F_2(x, g(x)) g'(x)\right] dx}_{④}$$

STOP HERE FOR A MOMENT!

Rewrite ③ as



$$③ = -\int_a^b \left[F_1(x, g(x)) - F_1(x, f(x))\right] dx$$

$$= -\int_a^b \left[F_1(x, y_{upper}) - F_1(x, y_{lower})\right] dx$$

From FTC I, for x fixed,

$$F_1(x, g(x)) - F_1(x, f(x)) = \int_{f(x)}^{g(x)} \frac{\partial F_1}{\partial y}(x, y)\, \partial y$$

Antiderivatives of $\frac{\partial F_1}{\partial y}$

Thus (3) $= \displaystyle\int_a^b \int_{y=f(x)}^{y=g(x)} \left[-\frac{\partial F_1}{\partial y}(x,y) \right] dy\, dx$

$$(3) = \iint_D \left[-\frac{\partial F_1}{\partial y}(x,y) \right] dA$$

OK, but we're still left with (4)! Nasty integral!

Solution: Set $F_2 = 0$!

In other words, we use linearity property of integrals:

$$\int_C \vec{F} \cdot d\vec{x} = \int_C \left[F_1(x,y)\hat{\imath} + F_2(x,y)\hat{\jmath} \right] \cdot d\vec{x}$$

$$= \int_C F_1(x,y)\,\hat{\imath} \cdot d\vec{x} + \int_C F_2(x,y)\,\hat{\jmath} \cdot d\vec{x}$$

This is (3).

We'll use other integration order to evaluate this integral, ie

$$\int_c^d \int_{x=h(y)}^{x=k(y)} \text{———}\; dx\, dy$$

↓ Result (Exercise)

$$\iint_D \frac{\partial F_1}{\partial y}\, dA$$

NET RESULT:

$$\boxed{\int_C \vec{F} \cdot d\vec{x} = \iint_D \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] dA}$$

■

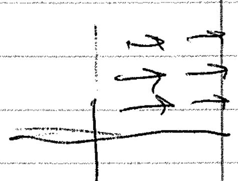
Another comment: You'll recall that if $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is conservative (or gradient), then

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

in which case the integrand on the right is zero, implying that the line integral of $\vec{F}$ over the closed curve C is zero - as we would expect! So there is consistency here with "conservativeness" of $\vec{F}$.

Let's now go back and examine the results of the earlier three examples in terms of Green's Theorem.

Example 1 $\quad \vec{F} = K\hat{\imath}$ $\quad$ Certainly the derivatives exist, everywhere:



$$\frac{\partial F_2}{\partial x} = 0 \qquad \frac{\partial F_1}{\partial y} = 0 \qquad \vec{F} \text{ is conservative}$$

This implies that all line integrals of $\vec{F}$ over simple closed curves in the plane are zero.

Example 2 $\quad \vec{F} = Kx\,\hat{i} + Ky\,\hat{j}$

Here $\quad \dfrac{\partial F_2}{\partial x} = 0 \quad \dfrac{\partial F_1}{\partial y} = 0 \quad \Rightarrow \vec{F}$ is conservative

From Green's Theorem, all circulation integrals are zero.

Example 3 $\quad \vec{F} = -Ky\,\hat{i} + Kx\,\hat{j}$

The derivatives exists everywhere:

$$\frac{\partial F_2}{\partial x} = K \qquad \frac{\partial F_1}{\partial y} = -K$$

Moreover $\quad \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = K - (-K) = 2K$

From Green's Theorem, for any closed curve C with interior region D,

$$\oint_C \vec{F} \cdot d\vec{x} = \iint_D \underbrace{\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]}_{2K} dA = 2K \iint_D dA$$

$$= 2K\, A(D)$$

where $A(D)$ is the area of region D.

For the circles C_R employed in Example 3,

$A(D) = \pi R^2 \Rightarrow \oint \vec{F} \cdot d\vec{x} = 2K\pi R^2$, in agreement with our computations