Lecture 8

Gradient fields/conservative forces

The treatment of gradient fields/conservative forces in the AMATH 231 Course Notes is rather brief. These lecture notes, which have been extracted from my MATH 227 (Fall 2010) lecture notes, should be used as the primary source of this material in this course.

Some sections of the text which follows were not presented in class but have been included for your information. They will be identified under the heading, "Extra material not covered in class."

Let us first consider the problem of motion in one dimension under the action of a force, which you saw in first-year Calculus and Physics. Suppose that a mass m is acted upon by a force $\mathbf{F}(x) = f(x)\mathbf{i}$ and that the motion of the mass, restricted to the x-axis, is determined by Newton's Law, i.e., $\mathbf{F} = m\mathbf{a}$. Equating components, we have

$$f = ma, \tag{1}$$

Here, $a(t) = x''(t)$, where $x(t)$ is the position of the mass.

Eq. (1) is a *differential equation* for the position function $x(t)$. In fact, it is a *second-order ordinary differential equation* (ODE) in $x(t)$, since the second derivative $x''(t)$ is involved. For $f(x)$ sufficiently "nice," which covers many common applications, there are mathematical theorems that guarantee the existence of solutions to this ODE. And as you will see in your course on differential equations (MATH 228), since Eq. (1) is a second-order ODE, if we impose two initial conditions on the solution, i.e., (i) $x(0) = x_0$ (initial position) and (ii) $x'(0) = v_0$ (initial velocity), we can extract a unique solution $x(t)$ that satisfies these conditions. For some special cases of $f(x)$, including the two examined below, the solutions to Eq. (1) may be obtained in closed form, i.e., in terms of simple functions. In other cases, one may have to rely on series expansions or numerical methods to generate approximate solutions.

Here, however, we are not interested in solving for $x(t)$ but rather in establishing a fundamental feature of the motion of the mass. First of all, we may define the *potential energy* associated with the

force $\mathbf{F}$ as follows,

$$V(x) = -\int_{x_0}^{x} f(s)ds, \tag{2}$$

where x_0 is a reference point. (Typically, we choose x_0 such that $V(x_0) = 0$, which often – but not always! – implies that $x_0 = 0$.) The physical interpretation of this definition is that the *potential energy $V(x)$ is the work done against the force $f(x)$ in moving the mass m from x_0 to x.*

Note, from the Fundamental Theorem of Calculus II, that

$$V'(x) = -f(x), \text{ or equivalently, } f(x) = -V'(x). \tag{3}$$

A force for which this relation holds, you may recall, is called a *conservative force.*

> Here we insert an important note: **Eq. (3) is to be considered as the definition of a one-dimensional conservative force**, i.e., one that may be written as the negative derivative of a potential energy function $V(x)$. The *conservation of energy*, which you know to be associated with a conservative force, is a **consequence** of Eq. (1), as we'll show after these examples.

In one dimension, all forces that are functions only of position are conservative. (A frictional force which, for a moving mass, is typically dependent upon the particle's velocity, is not conservative.) The total *mechanical energy* of the mass during its motion is given by the sum of its kinetic and potential energies, i.e.,

$$E(t) = \frac{1}{2}mv(t)^2 + V(x(t)). \tag{4}$$

We emphasize the dependence of the position $x(t)$ and velocity $v(t)$ on time t – the kinetic and potential energies need not be constant during the motion of the mass. (However, the energy $E(t)$ will be, but more on this later.)

Example 1: Near the surface of the earth, the gravitational force on a small object of mass m is well approximated by $\mathbf{F}(x) = -mg\mathbf{i}$, where x denotes the height of the object above the surface of the earth and the unit vector $\mathbf{i}$ points directly upward. Here, $f(x) = -mg$. For convenience, we let $x = 0$ be the surface of the earth so that the potential energy of the mass becomes

$$V(x) = -\int_{0}^{x} -mg \, ds = mgx. \tag{5}$$

Then for an object moving under the influence of gravity near the surface of the earth (for example, a ball that has been thrown upward, or dropped from a height $h > 0$), its total mechanical energy is

2

given by

$$E(t) = \frac{1}{2}mv(t)^2 + mgx(t). \tag{6}$$

Example 2: Recall Hooke's "Law" (it is NOT a law but an approximation) for the force exerted by a spring on a mass that is allowed to move only in one dimension: $\mathbf{F}(x) = -kx\mathbf{i}$, where k is the spring constant and x is the displacement from the equilibrium position $x = 0$. We also assume that there is no frictional force exerted on the mass. Here, $f(x) = -kx$. Once again we let $x = 0$ be reference point so that the potential energy function is defined as

$$V(x) = -\int_0^x -ks \, ds = \frac{1}{2}kx^2. \tag{7}$$

The total mechanical energy of the mass moving under the influence of the spring (oscillatory motion about $x = 0$) is given by

$$E(t) = \frac{1}{2}mv(t)^2 + \frac{1}{2}kx^2. \tag{8}$$

Conservation of energy

We now come to the following fundamental result, which accounts for the term "conservative" in "conservative forces." The motion of the mass m, when acted upon by a conservative force, is such that the *total mechanical energy $E(t)$* is conserved. The total mechanical energy is the sum of the kinetic energy (KE) and the potential energy (PE) of the mass. Along the trajectory $x(t)$, then, the energy of the mass is given by

$$E(t) = \frac{1}{2}mv(t)^2 + V(x(t)). \tag{9}$$

We now prove that $E(t)$ is constant along the trajectory $x(t)$ by showing that $E'(t) = 0$ for all time t. Let us differentiate (9) with respect to time t, using the Chain Rule (yes, the one of the most important methods in Calculus!):

$$
\begin{aligned}
E'(t) &= mv(t)v'(t) + V'(x(t))x'(t) \tag{10}\\
&= mv(t)a(t) - f(x(t))v(t)\\
&= v(t)[ma(t) - f(x(t))]\\
&= v(t) \cdot 0\\
&= 0,
\end{aligned}
$$

where we have used the result that $f = ma$ at all points/times on the trajectory $x(t)$. Since $E'(t) = 0$, it follows that $E(t)$ is constant in time. This constant value of the energy could be determined, for example, from the initial conditions, i.e., $x(0)$ and $x'(0) = v(0)$.

We now wish to extend this result to motion in higher dimensions, i.e. $\mathbf{R}^n$ for $n = 2, 3, \cdots$. The question is, "What is a conservative force $\mathbf{F}$ in $\mathbf{R}^2$ or $\mathbf{R}^3$?" Let us go back to the definition of a conservative force in one dimension, Eq. (3), and rewrite it in vector form as

$$\mathbf{F} = f(x)\mathbf{i}, \qquad \mathbf{F} = -\frac{\partial V}{\partial x}\mathbf{i}. \tag{11}$$

The expression on the right hand side is the one-dimensional form of a gradient vector. If we can generalize this result to $\mathbf{R}^n$ – and we can – then the definition of a conservative force $\mathbf{F}$ in $\mathbf{R}^n$ is one for which there exists a scalar-valued function $V : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = -\vec{\nabla}V. \tag{12}$$

Extra material not covered in class: Using conservation of energy to solve simple physical problems

In addition to being a a fundamental principle of classical mechanics, conservation of energy can often be an effective tool to analyze physical systems. In many cases, it is simpler, perhaps much simpler, to use conservation of energy in such analyses. You have probably encountered this idea in a first-year physics course, but it is important to recall this idea.

Example 1 revisited: Motion of a projectile near the surface of the earth

To illustrate, let us return to Example 1 above, specifically, to analyze motion under the influence of gravity near the surface of the earth. We consider the following well-known problem,

> A ball is thrown directly upwards with speed v_0. How high does it travel from the point at which it was released?

There are two ways to solve this problem:

1. **Method No. 1 – Solving for the particle's trajectory:** Given that the force $f(x) = -mg$, the equation of motion arising from Newton's law $f = ma$ is

$$mx''(t) = m\frac{d^2x}{dt^2} = -mg \qquad \rightarrow \qquad \frac{d^2x}{dt^2} = -g. \tag{13}$$

This is a very simple differential equation for $x(t)$ which can be integrated twice to obtain $x(t)$.

$$x(t) = -\frac{1}{2}gt^2 + c_1 t + c_2. \tag{14}$$

Imposing the initial conditions $x(0) = 0$ (reference point - the position from which the ball was launched) and $x'(0) = v_0$ (initial velocity), we obtain the well-known result,

$$x(t) = v_0 t - \frac{1}{2}gt^2. \tag{15}$$

From this equation, it follows that the velocity $v(t) = x'(t)$ is given by

$$v(t) = v_0 - gt, \tag{16}$$

which we would have obtained by integrating Eq. (13) only once. At the highest point of the ball's trajectory, the velocity is zero. From (16), we see that velocity $v(t) = 0$ at time $t^* = \frac{v_0}{g}$. Substitution into Eq. (15) yields that that

$$x(t^*) = \frac{v_0^2}{2g}. \tag{17}$$

This is the maximum height of the ball's trajectory.

2. **Method No. 2 – Using conservation of energy:** The total mechanical energy of the ball, given by Eq. (6), is constant in time. At $t = 0$, when the ball is launched, its mechanical energy is $E(0) = \frac{1}{2}mv_0^2$, i.e., all kinetic energy and zero potential energy. At the highest point of the ball's trajectory, which we shall denote as $x = h$, it has zero kinetic energy and potential energy mgh. By conservation of energy,

$$mgh = \frac{1}{2}mv_0^2, \tag{18}$$

from which it follows that $h = \frac{v_0^2}{2g}$, in agreement with the result of Method No. 1.

You can see that Method No. 2 is much simpler – we didn't have to solve for the ball's trajectory, then solve for the time at which it attains its maximum height, etc.. In more complicated problems, e.g., the motion of the earth around the sun (the so-called "Kepler problem" of motion under the influence of gravity, not near the surface of the earth!), it is quite tedious to solve for the position of the object of concern. Conservation of energy allows us to extract much information much more easily. Indeed, in problems involving more than two objects, e.g., sun-earth-moon, the trajectories of the masses may not be obtained in closed form.

Example 2 and its generalization: One-dimensional periodic motion in a "potential well"

We return to the mass-spring system of Example 2 above. Suppose that we place the mass m at a point $x_0 > 0$ – therefore stretching the spring – and then release it from rest. We expect the resulting motion of the mass to be oscillatory: The mass will move leftward, past the equilibrium point $x = 0$ and continue to move leftward, but slowing down until it stops and begins travelling to the right. It then moves past the equilibrium point $x = 0$ and then continues to its original starting point, $x = x_0$. Then it again begins its leftward journey, etc.. You are probably not surprised by the fact that the particle returned to its original starting point, since the total energy of the mass is conserved. You probably also know that when the mass is going through the equilibrium point $x = 0$, it possesses its maximum kinetic energy.

In fact, as we established above, the total mechanical energy of the mass, $E(t)$, is constant for all time $t \geq 0$, $t = 0$ being the time at which we released the mass. Recall that we determined the potential energy of this system (i.e., the potential energy stored in the spring) to be $V(x) = \dfrac{1}{2}kx^2$. Therefore, the initial mechanical energy of the system is $E(0) = \dfrac{1}{2}kx_0^2$. At any time $t \geq 0$, the mechanical energy of the system is

$$E(t) = \frac{1}{2}mv(t)^2 + \frac{1}{2}kx(t)^2 = \frac{1}{2}kx_0^2, \tag{19}$$

where $v(t) = x'(t)$ is the velocity of the mass.

Note that if we set $v(t) = 0$, we obtain $x(t) = x_0$, i.e., the original starting point. But another possibility is $x(t) = -x_0$. In other words, the mass executes oscillatory motion between the points $x = x_0$ and $x = -x_0$. In this case, the **amplitude** of the oscillation is clearly $A = x_0$.

This oscillatory motion may be represented graphically in a plot of potential energy vs. position, as shown in the figure below. In this figure, we have plotted the potential energy function $V(x) = \dfrac{1}{2}kx^2$. On the x-axis, we have also identified the points $\pm x_0$ representing the extremities of the trajectory $x(t)$ of the mass. These points are also called the **turning points** of the oscillatory trajectory – they are the points at which the mass "turns around" and reverses its motion. They are the points at which the instantaneous velocity of the mass is zero, implying that its potential energy is equal to its total energy $E(0)$.

In fact, we can solve Newton's equation of motion $f = ma$ for this case to obtain the trajectory $x(t)$ explicitly as a function of time. The result, which you may see in your course on differential

oscillatory motion in potential well
between turning points
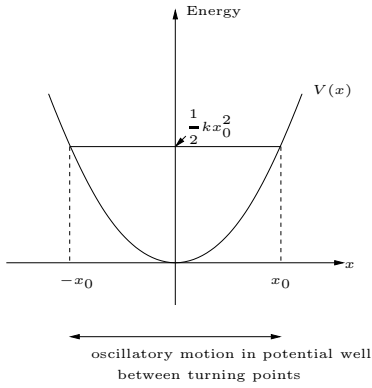
equations, is

$$x(t) = x_0 \cos(\omega t), \qquad \omega = \sqrt{\frac{k}{m}}. \tag{20}$$

Here, ω is the (angular) frequency of oscillation.

For more general one-dimensional potentials, however, we may not be able to solve for the trajectory $x(t)$ of a mass oscillating in a potential well. However, if we know the (constant) mechanical energy E of the mass, we may be able to solve for its turning points x_1 and x_2, shown schematically in the figure below. They are the points at which $V(x_1) = V(x_2) = E$. Moreover, the *period* of the oscillation may also be expressed as an integral involving the potential function $V(x)$, the total energy E and the turning points x_1 and x_2.

oscillatory motion in potential well
between turning points
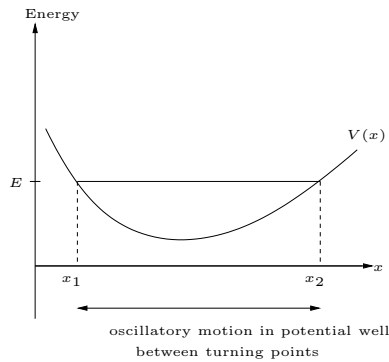
Gradient fields/conservative forces in physics (cont'd)

Definition: In physics, a force field $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ is conservative if there exists a scalar-valued **potential function** $V : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = -\vec{\nabla}V. \tag{21}$$

The above definition is quite specific to Physics because of the appearance of the minus sign on the LHS. (As you have seen earlier, this allows the total mechanical energy to be a sum of the kinetic energy and the potential energy V, not $-V$.) Mathematicians tend not to talk about **conservative** fields but rather **gradient** vector fields:

Definition: A vector field $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ is a **gradient vector field** if there exists a scalar-valued function $f : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\mathbf{F} = \vec{\nabla}f. \tag{22}$$

Of course, f and V are related by $V = -f$. We shall be using both definitions throughout this course. But when definite applications to Physics are considered, we shall be using the definition of conservative forces.

Note: The above definition includes the one-dimensional case studied in the previous lecture as a special case. Recall that in this case, we considered a force $\mathbf{F}(x) = f(x)\mathbf{i}$ that was dependent only upon the position variable x. By defining the associated potential function $V(x)$ as follows,

$$V(x) = \int_{x_0}^{x} f(s)\, dx, \tag{23}$$

we have, by the Fundamental Theorem of Calculus II that

$$f(x) = -V'(x) = -\frac{dV}{dx}. \tag{24}$$

Since we are working only in one dimension, this relation implies the equality of the following two vectors,

$$f(x)\,\mathbf{i} = -\frac{dV}{dx}\,\mathbf{i}. \tag{25}$$

The left-hand side is $\mathbf{F}(x)$. The right-hand side is the negative one-dimensional gradient of $V(x)$, so that we have

$$\mathbf{F} = -\vec{\nabla}V, \tag{26}$$

confirming that $\mathbf{F}$ is conservative.

Conservative forces and conservation of energy

The following represents one of the most important results of this course.

Suppose that a mass m is moving in $\mathbf{R}^n$ under the influence of a conservative force $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$, i.e., $\mathbf{F} = -\vec{\nabla} V$, where $V : \mathbf{R}^n \to \mathbf{R}$ (potential energy function). The trajectory $\mathbf{x}(t)$ of the mass is determined by Newton's Law, $\mathbf{F} = m\mathbf{a}$. Then the total mechanical energy E of the mass is constant over the trajectory. Here, E is defined as the usual sum of kinetic and potential energies of the mass, i.e.,

$$E(t) = \frac{1}{2}m\|\mathbf{v}(t)\|^2 + V(\mathbf{x}(t)). \tag{27}$$

It is important to note that the energy $E(t)$ is evaluated at points $\mathbf{x}(t)$ of the trajectory.

To prove this result, it will be useful to rewrite the total energy function as follows:

$$E(t) = \frac{1}{2}m\mathbf{v}(t) \cdot \mathbf{v}(t) + V(x_1(t), x_2(t), \cdots, x_n(t)). \tag{28}$$

We now differentiate both sides with respect to time t:

$$E'(t) = \frac{dE}{dt} = \frac{1}{2}m\frac{d}{dt}[\mathbf{v}(t) \cdot \mathbf{v}(t)] + \frac{d}{dt}V(x_1(t), x_2(t), \cdots, x_n(t)). \tag{29}$$

To determine the first derivative, we use the result:

$$\frac{d}{dt}\mathbf{a}(t) \cdot \mathbf{b}(t) = \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t). \tag{30}$$

(This is easily proved from the definition of the dot product:

$$\mathbf{a}(t) \cdot \mathbf{b}(t) = a_1(t)b_1(t) + a_2(t)b_2(t) + \cdots + a_n(t)b_n(t). \tag{31}$$

Simply use the product rule for differentiation and rearrange the results.)

Then

$$\begin{aligned} \frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t) &= \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) \\ &= 2\mathbf{v}'(t) \cdot \mathbf{v}(t) \\ &= 2\mathbf{a}(t) \cdot \mathbf{v}(t). \end{aligned} \tag{32}$$

The derivative involving V is computed using the Chain rule:

$$\begin{aligned} \frac{d}{dt}V(x_1(t), \cdots, x_n(t)) &= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} \\ &= \vec{\nabla} V \cdot \mathbf{v}. \end{aligned} \tag{33}$$

Putting these two results together gives

$$
\begin{aligned}
E'(t) &= \mathbf{v} \cdot [m\mathbf{a}(t)] + \mathbf{v}(t) \cdot \vec{\nabla} V &\qquad (34)\\
&= \mathbf{v}(t) \cdot [m\mathbf{a}(t) + \vec{\nabla} V]\\
&= \mathbf{v}(t) \cdot [m\mathbf{a}(t) - \mathbf{F}(\mathbf{x}(t))] \quad \text{(since } \mathbf{F} \text{ is conservative)}\\
&= \mathbf{v}(t) \cdot \mathbf{0} \quad \text{(trajectory determined by Newton's Law)}\\
&= 0.
\end{aligned}
$$

Therefore the energy $E(t)$ is constant along the trajectory of the mass.

Some examples of conservative forces in higher dimensions

Example 1: Consider the two-dimensional mass-spring system sketched below. (We are assuming that the mass is lying on a frictionless floor.)
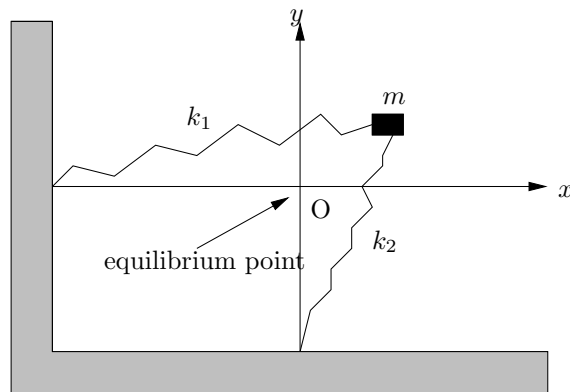
For small displacements from the equilibrium point $(0,0)$, the force exerted by the springs on the mass m is well-approximated as follows,

$$
\mathbf{F}(x,y) = -k_1 x \mathbf{i} - k_2 y \mathbf{j}. \qquad (35)
$$

(We are ignoring higher-order corrections. The exact result will involve square roots of sums involving squares of the coordinates x and y.) We now ask whether this force is conservative or not. In other words, does there exist a scalar-valued function $V(x,y)$ such that $\mathbf{F} = -\vec{\nabla} V$? In Lecture 7, we showed that a planar vector field,

$$
\mathbf{F}(x,y) = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j}, \qquad (36)
$$

is a gradient field, i.e., there exists an $f : \mathbf{R}^2 \to \mathbf{R}$, so that $\mathbf{F} = \vec{\nabla} f$, if the following condition exists,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{37}$$

Being a gradient field is equivalent to being a conservative field since $f - V$. So we simply check if Eq. (37) is satisfied by the field in (35):

$$\frac{\partial F_1}{\partial y} = 0, \qquad \frac{\partial F_2}{\partial x} = 0. \tag{38}$$

Therefore $\mathbf{F}$ is conservative and a potential energy function $V(x, y)$ exists. In component form, this means that

$$-k_1 x \mathbf{i} - k_2 y \mathbf{j} = -\frac{\partial V}{\partial x} \mathbf{i} - \frac{\partial V}{\partial y} \mathbf{j}. \tag{39}$$

By equating components, we are looking for a function $V(x, y)$ such that

$$\frac{\partial V}{\partial x} = k_1 x, \qquad \frac{\partial V}{\partial y} = k_2 y. \tag{40}$$

This problem is rather easy. By "inspection" or "educated guessing", we can come up with a suitable candidate:

$$V(x, y) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2, \tag{41}$$

which you can easily verify by differentiation. (We actually discussed a more systematic method of determining such scalar-valued functions in the previous lecture, i.e., Lecture 7, and will return to this idea shortly.)

Note that $V(x, y)$ is simply the sum of the "Hookean" potential energies of the two springs. Thus the force $\mathbf{F}$ is conservative. This implies that the total energy of the mass is constant during any motion (or even if it is lying at rest at the equilibrium point!). Its total energy will be given by

$$E(t) = \frac{1}{2} m v(t)^2 + V(x(t), y(t)), \tag{42}$$

where $v(t) = \| \mathbf{v}(t) \|$ is the speed of the mass at time t. If we drop the reference to time t and let $\mathbf{v} = (v_1, v_2)$, then the total mechanical energy is given by

$$\begin{aligned} E &= \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 + \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x^2 \tag{43} \\ &= \left[\frac{1}{2} m v_1^2 + \frac{1}{2} k_1 x^2 \right] + \left[\frac{1}{2} m v_2^2 + \frac{1}{2} k_2 y^2 \right]. \end{aligned}$$

$$\tag{44}$$

In other words, the total energy can be viewed as the the sum of the mechanical energies in the x and y directions where each spring acts independently along a specific direction.

Before moving on, let us return to the potential energy function $V(x, y)$ for this problem, as given in Eq. (41). A sample graph of this function is sketched in the figure below. This is a two-dimensional "potential well", as opposed to the one-dimensional potential well sketched in the previous lecture. And the minimum of this potential well, $(0, 0)$, is the equilibrium point of the system – the point at which the net force on the mass is zero.
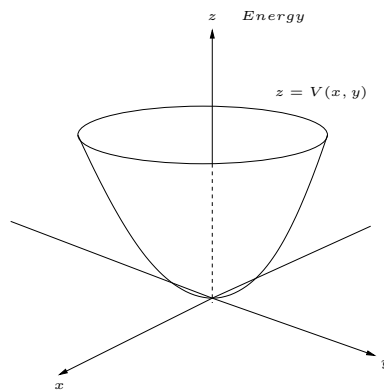
If the mass has a total mechanical energy $E_0 \geq 0$ at some time t, then its total energy $E(t) = E_0$ for all time t. The "classical turning points" of its oscillation, that is, the points at which it will change its direction of motion, consist of all points (x, y) such that $V(x, y) = E_0$. Recall that for one-dimensional motion, there will be only two such turning points at which the mass reverses its direction. But in two dimensions, the turning points form a curve. In this case, the curve is the ellipse in the xy-plane given by

$$V(x, y) = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 = E_0. \tag{45}$$

At these points, the mass has zero kinetic energy – all of its energy has been converted into potential energy.

Example 2: In Problem Set No. 1, you derived the following result in $\mathbf{R}^3$: Given the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and its magnitude $r = [x^2 + y^2 + z^2]^{1/2}$, then for any n,

$$\vec{\nabla} r^n = nr^{n-2}\mathbf{r}. \tag{46}$$

In the special case $n = -1$, we have

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r}. \tag{47}$$

Note that for any constant K,

$$\vec{\nabla}\left(\frac{K}{r}\right) = -\frac{K}{r^3}\mathbf{r}. \tag{48}$$

The term on the right side resembles the form of vector force fields that are associated with either a charge or mass situated at the origin.

1. **Case 1: Electrostatic fields**

   Recall that the electrostatic field (that is, the force per unit charge) $\mathbf{E}(\mathbf{r})$ that is generated by the presence of a charge Q at the origin is given by

   $$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r}. \tag{49}$$

   Note that if we set

   $$K = -\frac{Q}{4\pi\epsilon_0}, \tag{50}$$

   then Eq. (48) becomes

   $$\vec{\nabla}\left(-\frac{Q}{4\pi\epsilon_0 r}\right) = \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r} = \mathbf{E}(\mathbf{r}). \tag{51}$$

   This implies that the electrostatic field $\mathbf{E}$ is conservative, i.e.

   $$\mathbf{E} = -\vec{\nabla}\Phi = \vec{\nabla}(-\Phi), \qquad \text{where} \ \ \Phi(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r}. \tag{52}$$

   $\Phi(\mathbf{r})$ is the *electrostatic potential* associated with the electrostatic field $\mathbf{E}(\mathbf{r})$. In Cartesian coordinates, it is given by

   $$\Phi(x, y, z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2 + y^2 + z^2}}. \tag{53}$$

   The level sets of Φ are spheres in $\mathbf{R}^3$ that are centered at the origin $(0, 0, 0)$. The value of the function $\Phi(\mathbf{r})$ decreases as we move away from the origin. Once again, this is reflected in the fact that its gradient vectors point inward. And since $\mathbf{E}$ is the *negative* of these gradient vectors, $\mathbf{E}$ points outward.

   You may be more familiar with the *electrostatic force* $\mathbf{F}(\mathbf{r})$ exerted by a charge Q at $(0, 0, 0)$ on a test charge q that is situated at $\mathbf{r}$. It will be given by

   $$\mathbf{F}(\mathbf{r}) = \frac{Qq}{4\pi\epsilon_0 r^3}\mathbf{r} = q\mathbf{E}(\mathbf{r}). \tag{54}$$

The *potential energy* of the charge q at $\mathbf{r}$ is then

$$V(\mathbf{r}) = \frac{Qq}{4\pi\epsilon_0 r} = q\Phi(\mathbf{r}). \tag{55}$$

These quantities are related, respectively, to the field quantitites $\mathbf{E}$ and Φ by the test charge constant q so that

$$\mathbf{F}(\mathbf{r}) = -\vec{\nabla}V(\mathbf{r}). \tag{56}$$

In other words, the electrostatic force field $\mathbf{F}$ is conservative.

2. **Case 2: Gravitational fields**

The above method is also easily applied to the gravitational field $\mathbf{G}(\mathbf{r})$ due to a mass M situated at the origin $(0, 0, 0)$. Here,

$$\mathbf{G}(\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r}. \tag{57}$$

In this case, we choose

$$K = GM \tag{58}$$

so that Eq. (48) becomes

$$\vec{\nabla}\left(\frac{GM}{r}\right) = -\frac{GM}{r^3}\mathbf{r} = \mathbf{G}(\mathbf{r}). \tag{59}$$

This implies that

$$\mathbf{G} = -\vec{\nabla}\Phi = \vec{\nabla}(-\Phi), \quad \text{where} \quad \Phi(\mathbf{r}) = -\frac{GM}{r}. \tag{60}$$

$\Phi(\mathbf{r})$ is now the *gravitational potential* associated with the gravitational field $\mathbf{G}(\mathbf{r})$.

Once again, you are more familiar with the *gravitational force* $\mathbf{F}(\mathbf{r})$ exerted by a mass M at $(0, 0, 0)$ on a mass m that is situated at $\mathbf{r}$. It will be given by

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\mathbf{r} = m\mathbf{G}(\mathbf{r}). \tag{61}$$

The *potential energy* of the mass m at $\mathbf{r}$ is then

$$V(\mathbf{r}) = -\frac{GMm}{r} = m\Phi(\mathbf{r}). \tag{62}$$

These quantities are related, respectively, to the field quantitites $\mathbf{E}$ and Φ by the mass m so that

$$\mathbf{F}(\mathbf{r}) = -\vec{\nabla}V(\mathbf{r}). \tag{63}$$

In other words, the gravitational force field $\mathbf{F}$ is conservative.

14

Conservative forces in physics (cont'd)

Determining whether or not a force is conservative

We have just examined some examples of conservative forces in $\mathbf{R^2}$ and $\mathbf{R^3}$. It is now natural address the following question:

Suppose that we are given a force $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$.

1. How do we determine whether or not $\mathbf{F}$ is conservative?

2. And if $\mathbf{F}$ is conservative, how do we find the potential $V : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = -\vec{\nabla}V. \tag{64}$$

Fortunately, we have answered all of these questions in Lecture 7, where the potential function V was replaced by the scalar-valued function f. Recall that $F : \mathbf{R}^n \to \mathbf{R}^n$ is said to be

1. a **gradient field** if there exists a scalar-valued function $f : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = \vec{\nabla}f \qquad \text{(mathematics)}, \tag{65}$$

2. a **conservative field** if there exists a scalar-valued function $V : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = -\vec{\nabla}V \qquad \text{(physics)}. \tag{66}$$

Therefore, we have the relationship

$$f = -V \quad \text{or} \quad V = -f \,. \tag{67}$$

Actually, a little more precisely, f and V are related as follows,

$$f = -V + C \quad \text{or} \quad V = -f + D \,, \tag{68}$$

where C and D are real-valued constants. (The gradient operator $\vec{\nabla}$ removes constants.)

Because Questions (a) and (b) above were answered in Lecture 7, we simply summarize the results below and perform a couple of sample computations. There is really nothing new here – the purpose is simply to get you acquainted with working with the potential energy function v instead of the scalar

function f.

Case 1: One-dimensional vector fields $(n = 1)$

This case is straightforward. Any one dimensional force $\mathbf{F} = f(x)\mathbf{i}$ that is a function only of position x is a conservative force. Recall that its associated potential function is defined by

$$V(x) = -\int_{x_0}^{x} f(s)\, ds, \tag{69}$$

where x_0 is a chosen reference point for which $V(0) = 0$.

Case 2: Two-dimensional vector fields $(n = 2)$

We now deal with forces of the form

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}. \tag{70}$$

If $\mathbf{F}$ is conservative, then it is a gradient field so that the following conditions must be satisfied:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{71}$$

This is a necessary condition on F_1 and F_2 for $\mathbf{F}$ to be conservative.

We examined the some of the following examples in Lecture 7, but did not consider them as forces:

Example 1: The force $\mathbf{F} = (3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j}$. Here $F_1(x, y) = 3x^2 - 3y^2$ and $F_2(x, y) = -6xy$. We compute:

$$\frac{\partial F_1}{\partial y} = -6y, \qquad \frac{\partial F_2}{\partial x} = -6y. \tag{72}$$

Therefore $\mathbf{F}$ is conservative.

Example 2: A slight modification of the force in Example 1: $\mathbf{F} = (3x^2 - 3y^2)\mathbf{i} + (4x - 6xy)\mathbf{j}$. Here $F_1(x, y) = 3x^2 - 3y^2$ and $F_2(x, y) = 4x - 6xy$. We compute:

$$\frac{\partial F_1}{\partial y} = -6y, \qquad \frac{\partial F_2}{\partial x} = 4 - 6y. \tag{73}$$

Therefore $\mathbf{F}$ is not conservative.

Example 3: The force $\mathbf{F} = (4x^2 - 4y^2)\mathbf{i} + (8xy - \ln y)\mathbf{j}$. Here $F_1(x,y) = 4x^2 - 4y^2$ and $F_2(x,y) = 8xy - \ln y$. We compute:

$$\frac{\partial F_1}{\partial y} = -8y, \qquad \frac{\partial F_2}{\partial x} = 8y. \tag{74}$$

Therefore $\mathbf{F}$ is not conservative. (It is insufficient that the above partial derivatives are equal on the line $y = 0$. In fact, the original vector field $\mathbf{F}(x,y)$ is not even defined at $y = 0$!)

Now suppose that we have determined that a force $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ is conservative. How can we find the associated potential function $V(x,y)$ such that

$$\mathbf{F} = -\vec{\nabla}V \ ? \tag{75}$$

Once again, the method was described in Lecture 7 but we'll go through the details again for Example 1 above. Since that force is conservative, it follows that there exists a $V(x,y)$ such that

$$(3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j}. \tag{76}$$

Equating components, we have the relations

$$\frac{\partial V}{\partial x} = -3x^2 + 3y^2 \quad (a), \qquad \frac{\partial V}{\partial y} = 6xy \quad (b). \tag{77}$$

From (a), we are looking for a function $V(x,y)$ which, when differentiated partially with respect to x, gives $-3x^2 + 3y^2$. We can "work backwards" by antidifferentiating with respect to x, keeping y fixed:

$$V(x,y) = \int (-3x^2 + 3y^2) \, \partial x = -x^3 + 3xy^2 + g(y). \quad (c) \tag{78}$$

We have used the notation ∂x to emphasize this partial antidifferentiation respect to x. Note also that the "constant of integration" is an unknown function $g(y)$ since any constant or function of y will be eliminated upon partial differentiation with respect to x. Obviously, we must determine $g(y)$. If we differentiate (c) partially with respect to y:

$$\frac{\partial V}{\partial y} = 6xy + g'(y), \quad (d) \tag{79}$$

Note that the derivative of g with respect to y is a normal derivative since g was assumed to be a function *only* of y and not x. We now compare (d) and (b):

$$g'(y) = 0, \tag{80}$$

17

which implies that $g(y) = C$ a constant. The final result is

$$V(x, y) = -x^3 + 3xy^2 + C. \tag{81}$$

This is a one-parameter family of potential functions associated with the conservative force $\mathbf{F}$. You should always check your result:

$$-\frac{\partial U}{\partial x} = 3x^2 - 3y^2 = F_1, \qquad -\frac{\partial U}{\partial y} = -6xy = F_2, \tag{82}$$

so our result is correct.

Note that we used both pieces of information (a) and (b) above. We could have started, however, with (b), partially integrated with respect to y, etc..

Note: We actually used this example in Lecture 7, showing that it was a gradient field, and then computing the scalar valued function $f : \mathbf{R}^2 \to \mathbf{R}$ so that $\mathbf{F} = \vec{\nabla}$. Our result was

$$f(x, y) = x^3 - 3xy^2 + \text{constant} \tag{83}$$

which agrees with our result for $V(x, y)$ since, as we've already discussed, $f = -V$, up to a constant.

The above procedure might seem quite similar to something that you have seen in your course on differential equations (e.g., MATH 228). There, you encountered differentials of the form,

$$M(x, y)dx + N(x, y)dy. \tag{84}$$

If $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, then the above is an exact differential, i.e., there exists a function $V(x, y)$ such that

$$dV = M(x, y)dx + N(x, y)dy. \tag{85}$$

You can then find $V(x, y)$ by partial integration, in the same way as we found the potential $V(x, y)$ above.

Case 3: Three-dimensional vector fields ($n = 3$): We now deal with forces of the form

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}. \tag{86}$$

If $\mathbf{F}$ is conservative, then there exists a scalar-valued potential function $V(x, y, z)$ such that $\mathbf{F} = -\vec{\nabla}V$, i.e.,

$$\mathbf{F}(x, y, z) = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}. \tag{87}$$

We simply state the results derived in Lecture No. 6: The relations that must be satisfied by the three components of a conservative force $\mathbf{F}$ in $\mathbf{R}^3$ are:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{88}$$

As mentioned before, these relations, which must hold *simultaneously*, look quite complicated. Later, we'll see that there is a rather simple formula that compactly contains these results.

Finally, how would one determine the potential $U(x, y, z)$ associated with a conservative force $\mathbf{F}$ in $\mathbf{R}^3$? The answer: By tedious, systematic partial integration along the lines of what was done for $\mathbf{R}^2$ earlier. Very fortunately, we have already encountered the most important conservative forces in $\mathbf{R}^3$: forces of the form

$$\mathbf{F}(\mathbf{r}) = -\frac{K}{r^3}\mathbf{r} \tag{89}$$

for which the associated potential function is

$$V(\mathbf{r}) = -\frac{K}{r}. \tag{90}$$

The two important cases examined earlier are:

1. $K = GMm$: gravitational force exerted by mass M at (0,0,0) on mass m at $\mathbf{g}$,

2. $K = -\dfrac{Qq}{4\pi\epsilon_0}$: electrostatic force exerted by charge Q at (0,0,0) on charge q at $\mathbf{g}$.

Lecture 9

Line (Path) integrals: Integration along curves in $\mathbf{R}^n$

(Relevant section from AMATH 231 Course Notes: Chapter 2, Section 2.1)

In this section, we shall be integrating scalar-valued functions $f(x, y, z)$ and vector-valued functions $\mathbf{F}(x, y, z)$ along *curves* in $\mathbf{R}^3$. The motivation for doing these things will become clear as the discussion progresses.

In what follows, it will be important to keep in mind that we shall be working on curves C in $\mathbf{R}^3$ that will be parametrized in terms of a single variable as follows, Thus

$$\mathbf{x}(t) = \mathbf{g}(t) = (x(t), y(t), z(t)), \quad a \le t \le b. \tag{91}$$

The notation "$\mathbf{g}(t)$" conforms to the notation used in the textbook (AMATH 231 Course Notes). In most applications, the curves will be either

1. "C^1 paths," meaning that the components have continuous derivatives for all $t \in [a, b]$.

   Example: The curve $\mathbf{g}(t) = (x(t), y(t), z(t)) = (\cos t, \sin t, 2t)$ for $t \in [0, \pi]$

2. "Piecewise C^1 paths," meaning that the interval $[a, b]$ can be broken down into a union of subintervals $[t_{k-1}, t_k]$, $k = 1, 2, \cdots, n$, where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, and the path $\mathbf{g}(t)$ is C^1 on each subinterval. (The derivatives may not necessarily exist at the points t_k.

   Example: The square in $\mathbf{R}^2$ with corners (0,0), (1,0), (1,1) and (0,1), parametrized as

$$\mathbf{g}(t) = (x(t), y(t)) = \begin{cases} (t, 0), & t \in [0, 1], \\ (1, t), & t \in [0, 1], \\ (1 - t, 1), & t \in [0, 1], \\ (0, 1 - t), & t \in [0, 1], \end{cases} \tag{92}$$

Note: From the above example, it's fine to use the same parameter interval $t \in [0, 1]$ to generate each of the component curves. If you wanted, however, to generate the entire curve in a continuous way, for example, to describe the motion of a particle over the entire square, then you would have to adjust the parameter intervals accordingly, i.e., $t \in [0, 1]$ over the bottom line segment, $t \in [1, 2]$ over the second segment, etc..

A brief review of the Riemann integral of $f(x)$ over the (straight) interval $[a, b]$.

Let us briefly recall the working definition of the Riemannian integral of the single-variable function $f(x)$ over the interval $[a, b]$:

1. We partition the interval $I = [a, b]$ into n subintervals of length $\Delta x = \dfrac{b - a}{n}$ using the partition points $x_k = a + k\Delta x$, $k = 0, 1, 2, \cdots, n$.

2. From each subinterval $I_k = [x_{k-1}, x_k]$, pick a sample point x_k^* and evaluate $f(x_k^*)$.

3. Form the Riemann sum
$$S_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Now let $n \to \infty$, implying that $\Delta x \to 0$. If $f(x)$ is sufficiently "nice" (i.e., piecewise continuous), then
$$\lim_{n \to \infty} S_n = S = \int_a^b f(x)\, dx$$

Special case: When $f(x) = 1$, the Riemann integral is
$$\int_a^b dx = b - a = L\,,$$

the length of the interval $[a, b]$.

Some simple applications of the Riemann integral to physics

1. **Total mass of a wire:** The interval $[a, b]$ could represent a very thin and straight wire. Suppose that $f(x)$ represents the *lineal mass density* of the wire, the (limiting) amount of mass per unit length at a point $x \in [a, b]$. (The contribution over the (small) cross sectional area is taken into account in the construction of f.) In the case that the wire is *homogeneous*, then $f(x) = C$, for all $x \in [a, b]$, where C is a constant. The total mass of the wire is then simply

$$M = C(b - a) \qquad (\text{lineal density} \times \text{length of wire})\,.$$

If $f(x)$ is not constant, we must resort to a Riemann integral approach to determine the total mass M. In the above procedure, the partitioning of the interval I into subintervals I_k breaks up the wire into n pieces of length Δx. The mass of each piece Δm_k located in the subinterval $I_k = [x_{k-1}, x_k]$ is approximated as
$$\Delta m_k \approx f(x_k^*)\Delta x.$$

In other words, we are making the approximation that the lineal density $f(x) \approx f(x_k^*)$, a constant, over the subinterval $I_k = [x_{k-1}, x_k]$. The total mass of the wire is then

$$M = \sum_{k=1}^{n} \Delta m_k \approx \sum_{k=1}^{n} f(x_k^*)\Delta x = S_n.$$

In the limit $n \to \infty$, the total mass of the wire becomes

$$M = \int_a^b f(x)\, dx.$$

Note: The function $f(x)$ could have represented the *lineal charge density* along the wire, the amount of charge per unit length. Then "mass" above would be replaced by "charge" so that the total charge Q in the wire is given by

$$Q = \int_a^b f(x)\, dx.$$

2. **Total work done by a nonconstant force:** Suppose that a force $\mathbf{F}(x) = f(x)\mathbf{i}$ is acting on a mass m, resulting in the displacement of the mass from position $x = a$ to $x = b$. We wish to compute the total work done by the force. In the case that the force is constant over $[a, b]$, i.e., $\mathbf{F}(x) = F$, where F is constant, then the total work W done by the force is

$$W = F(b - a) \qquad \text{(force)} \times \text{displacement}.$$

If the force $f(x)$ is not constant over $[a, b]$, then we must once again resort to a Riemann integral approach to determine the total mass M. Proceeding in the same way as above, we form the partition of the interval $[a, b]$ into n subintervals I_k of length Δx. We then approximate the work ΔW_k done by the force in moving the mass from x_{k-1} to x_k in subinterval I_k by

$$\Delta W_k \approx f(x_k^*)\Delta x.$$

In other words, we are making the approximation that the force $f(x)$ exerted on the mass over the interval $[x_{k-1}, x_k]$ is constant, i.e., $f(x) \approx f(x_k^*)$. The total work W done by the force is

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k=1}^{n} f(x_k^*)\Delta x = S_n.$$

In the limit $n \to \infty$, the total work becomes

$$W = \int_a^b f(x)\, dx.$$

22

It is important to keep these simple examples in mind. They represent the "Spirit of Calculus" which will be used to solve more complicated problems.

As a preview of things to come, suppose that the wire in Example 1 above is no longer straight but has the shape of a curve C in $\mathbf{R}^3$. If we know the lineal mass/charge density function $f(x, y, z)$ along the wire, how can we compute the total mass/charge of the wire?

And for an even more complicated problem, suppose that a nonconstant force $\mathbf{F}(x, y, z)$ is acting on a mass m, causing it to move along a curve $\mathbf{x}(t) = \mathbf{g}(t)$ in $\mathbf{R}^3$. What is the total work done by the force on the mass?

These are motivations for the study of path integrals of scalar and vector-valued functions.

Line (Path) integrals of scalar-valued functions

Given a curve C with endpoints P and Q in $\mathbf{R}^3$. Now suppose that there is a scalar valued function $f : \mathbf{R}^3 \to \mathbf{R}$ that is defined at all points on the curve C. For example, the curve C could represent a thin wire and $f(x, y, z)$ could represent the lineal charge density at a point (x, y, z) on the wire.

Let us now introduce points $P_0 = P$, P_1, P_2, ... P_{n-1}, $P_n = Q$ on the curve as shown below.
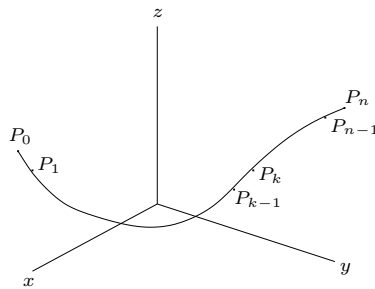
Let Δs_k denote the length of each "subcurve" C_k with endpoints P_{k-1} and P_k. Now on each subcurve C_k pick a sample point (x_k^*, y_k^*, z_k^*) and compute the product

$$f(x_k^*, y_k^*, z_k^*)\Delta s_k. \tag{93}$$

If f were the lineal charge density, then the above amount would be the approximate amount of charge Δq_k on the subcurve C_k. We have once again made the approximation that the lineal density f is constant over the subcurve.

23

Now form the sum

$$S_n = \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta s_k. \tag{94}$$

If f is "sufficiently nice," i.e piecewise continuous, and C as well is "sufficiently nice," i.e. piecewise C^1, then the $\lim_{n\to\infty} S_n$ exists, provided that all the $\Delta s_k \to 0$. We denote this limit as

$$\int_C f \, ds \tag{95}$$

and call it the "line integral of f over curve C". If $f(x, y, z) = 1$ on curve C, then

$$\int_C f \, ds = \int_C ds = L, \text{ the length of curve } C. \tag{96}$$

We have already discussed the computation of arclengths of curves in this course. The extension to compute line integrals of functions is quite straightforward.

Let's start at the beginning, however, and proceed in the same way that we computed arclengths of curves. Recall that we began by finding a parametrization of the curve C, i.e.,

$$\mathbf{x}(t) = \mathbf{g}(t) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Now divide the interval $[a, b]$ into n subintervals of length $\Delta t = (b-a)/n$ via the partition points $t_k = a + k\Delta t$, $k = 0, 1, 2, \cdots, n$. We now let the points P_k introduced earlier correspond to the points $\mathbf{g}(t_k) = (x(t_k), y(t_k), z(t_k))$.

From each subinterval $[t_{k-1}, t_k]$, pick a sample point t_k^* so that

$$(x_k^*, y_k^*, z_k^*) = \mathbf{g}(t_k^*) = (x(t_k^*), y(t_k^*), z(t_k^*))$$

is the sample point at which we evaluate $f(x, y, z)$. We now form the sum

$$\begin{aligned} T_n &= \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta s_k \\ &= \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*), z(t_k^*)) \Delta s_k. \end{aligned} \tag{97}$$

Recall that in the arclength problem, we had to determine the lengths Δs_k of the subcurves C_k. We approximated these lengths by the lengths of the straight line segments $\overline{P_{k-1}P_k}$. Then, by Pythagoras,

$$\Delta s_k \approx \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}, \tag{98}$$

24

where

$$\Delta x_k = x(t_k) - x(t_{k-1}), \quad \Delta y_k = y(t_k) - y(t_{k-1}), \quad \Delta z_k = z(t_k) - z(t_{k-1}). \tag{99}$$

We still need to bring the parameter t into the expression for Δs_k. We do this by multiplying and dividing by Δt:

$$\Delta s_k \approx \sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2 + \left(\frac{\Delta z_k}{\Delta t}\right)^2} \, \Delta t \, . \tag{100}$$

Now insert this term into (97) to give

$$T_n = \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*), z(t_k^*)) \sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2 + \left(\frac{\Delta z_k}{\Delta t}\right)^2} \, \Delta t. \tag{101}$$

This is a Riemann sum. In the limit $n \to \infty$, we have $\Delta t \to 0$ so that the points t_{k-1} and t_k approach each other. The square root term is an approximation to the magnitude of the velocity, i.e., the speed, associated with the parametrization. As $\Delta t \to 0$, it becomes better and better approximated by the becomes the speed $\|\mathbf{g}(t)\|$ evaluated at the sample point t_k^* that lies between t_{k-1} and t_k, i.e.,

$$\sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2 + \left(\frac{\Delta z_k}{\Delta t}\right)^2} \, \Delta t \approx \|\mathbf{g}'(t_k)\| \Delta t \, . \tag{102}$$

The Riemann sum T_n then becomes approximated by the Riemann sum,

$$S_n \approx \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*), z(t_k^*)) \|\mathbf{g}'(t_k^*)\| \Delta t \, . \tag{103}$$

Once again, for f and C "sufficiently nice," the limit of the S_n exists and is the desired line integral:

$$\lim_{n \to \infty} S_n = \int_C f \, ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt. \tag{104}$$

The final integral on the right is an integration over the parameter t. There are a couple of important points regarding this integral:

1. The term $f(\mathbf{g}(t)) = f(x(t), y(t), z(t))$ represents the function f being evaluated *over points on the curve C*.

2. The product $\|\mathbf{g}'(t)\| dt = ds$, the infinitesimal element of arclength associated with the parametrization $\mathbf{g}(t)$. It is simply the statement that the infinitesimal displacement along the curve ds is given by the product of the instantaneous speed $\|\mathbf{v}(t)\| = \|\mathbf{g}'(t)\|$ with the element of time dt, i.e.,

$$ds = \|\mathbf{g}'(t)\| \, dt \, . \tag{105}$$

25

Special case: When $f(x, y, z) = 1$, the line integral over curve C reduces to

$$\int_C f \, ds = \int_C ds = \int_a^b \|\mathbf{g}'(t)\| \, dt = \int_a^b \|\mathbf{v}(t)\| \, dt = L, \quad \text{the length of curve } C. \tag{106}$$

We derived this result earlier in the course. In fact, most of the derivation of the above line integration is the same as what you saw earlier – the only modification is the multiplication of the lengths Δs_k of the subcurves by sample values of the function f over those subcurves.

Some examples of line integrals of scalar fields

In what follows we shall compute $\int_C f(x, y) \, ds$ on various curves in $\mathbf{R}^2$ that start at (0,0) and end at (1,1). We shall use the function $f(x, y) = x^2 + y^2$.

1. C is the straight line from (0,0) to (1,1), parametrized as

$$x(t) = t, \quad y(t) = t, \quad 0 \leq t \leq 1.$$
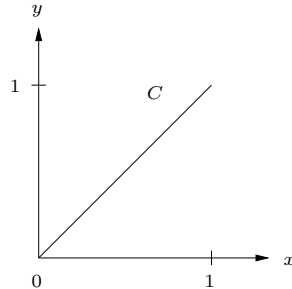
In compact form, the parametrization is $\mathbf{g}(t) = (t, t)$. The velocity is $\mathbf{g}'(t) = (1, 1)$ so that the speed is $\|\mathbf{g}'(t)\| = \sqrt{2}$. The element of arclength is given by $ds = \sqrt{2} dt$.

We must still evaluate the function $f(x, y)$ over the curve:

$$f(\mathbf{g}(t)) = f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + t^2 = 2t^2.$$

Therefore the desired line integral becomes

$$\int_C f \, ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt = \int_0^1 2t^2 \sqrt{2} \, dt = 2\sqrt{2} \int_0^1 t^2 dt = \frac{2\sqrt{2}}{3}. \tag{107}$$

2. Let us now compute the line integral over the straight line from (0,0) to (1,1) but now parametrized as

$$x(t) = t^2, \quad y(t) = t^2, \quad 0 \leq t \leq 1.$$

This means that $\mathbf{g}(t) = (t^2, t^2)$ so that the velocity is $\mathbf{g}'(t) = (2t, 2t)$ and the speed is

$$\|\mathbf{g}'(t)\| = \sqrt{4t^2 + 4t^2} = 2\sqrt{2}t.$$

Thus, $ds = 2\sqrt{2}t\, dt$.

Now evaluate the function $f(x, y)$ over the curve:

$$f(\mathbf{g}(t)) = f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^4 + t^4 = 2t^4.$$
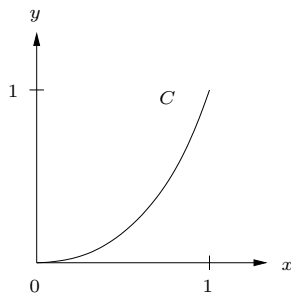
The line integral becomes

$$\int_C f\, ds = \int_a^b f(\mathbf{g}(t))\|\mathbf{g}'(t)\|\, dt = \int_0^1 2t^4 2\sqrt{2}t\, dt = 4\sqrt{2}\int_0^1 t^5 dt = \frac{2\sqrt{2}}{3}. \qquad (108)$$

Note that we arrive at the same result as in 1. above. This is not a conclusive proof but we do state the general result that a line integral is independent of the parametrization used to evaluate the line integral. In retrospect, this is a very natural/desirable result. After all, the parametrization is only a book-keeping method of keeping track of points that lie on the curve and how they are visited. Changing parametrizations can change the velocity, which determines how short or long a time is spent over a particular interval, based on the parametrization. The element of arclength $ds = \|\mathbf{g}'(t)\|dt$ performs the necessary book-keeping. (We'll prove this shortly.)

3. We now consider the following curve from (0,0) to (1,1),

$$x(t) = t, \quad y(t) = t^2, \quad 0 \le t \le 1.$$

This curve lies on the parabola $y = x^2$.

Then $\mathbf{g}(t) = (t, t^2)$, implying that $\mathbf{g}'(t) = (1, 2t)$ so that $\|\mathbf{g}'(t)\| = \sqrt{1 + 4t^2}$. Thus, the infinitesimal arclength is given by $ds = \sqrt{1 + 4t^2}\, dt$.

Evaluating the function $f(x, y)$ over the curve:

$$f(\mathbf{g}(t)) = f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + t^4.$$

The line integral becomes

$$\int_C f \, ds = \int_a^b f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| \, dt = \int_0^1 (t^2 + t^4)\sqrt{1 + 4t^2} \, dt$$

$$= \frac{349}{768}\sqrt{5} - \frac{7}{512}\ln\left(1 + \frac{1}{2}\sqrt{5}\right) - \frac{7}{256}\ln 2$$

$$\approx 0.996. \tag{109}$$

This result is different from those of Examples 1 and 2 because of the different path taken.

4. Let us now consider the following example in $\mathbf{R}^3$: The curve $\mathbf{g}(t) = (t, 2t^2, 3t^2)$ for $t \in [0, 1]$. The velocity is $\mathbf{g}'(t) = (1, 4t, 6t)$ so that the speed is $\|\mathbf{g}'(t)\| = \sqrt{1 + 52t^2}$.

Consider the function $f(x, y, z) = x^2 + xyz + z^3$. Then, evaluating f over the curve:

$$f(x(t), y(t), z(t)) = t^2 + 6t^5 + 27t^6.$$

Thus,

$$\int_C f \, ds = \int_0^1 f(\mathbf{g}(t)) \|\mathbf{g}'(t)\| dt \tag{110}$$

$$= \int_0^1 (t^2 + 6t^5 + 27t^6)\sqrt{1 + 52t^2} dt$$

The approxmate numerical value of this parameter is 32.717 (Maple).

Applications of line (path) integrals of scalar-valued functions

1. The average value of a function $f(x, y, z)$ over a curve C is given by

$$\bar{f}_C = \frac{\int_C f \, ds}{\int_C ds} = \frac{\int_C f \, ds}{L}, \tag{111}$$

where L is the length of the curve C. This is a generalization of the definition of the average value of a function $f(x)$ over the interval $[a, b]$.

Going back to Examples 1 and 2 of this section, we computed the line integral of $f(x, y) = x^2 + y^2$ over the straight line segment joining $(0, 0)$ to $(1, 1)$ as $2\sqrt{2}/3$. The length of line segment is $\sqrt{2}$. Therefore the average value of $f(x, y)$ on this line segment is

$$\bar{f}_C = \frac{2\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{2}{3}. \tag{112}$$

28

Note: The average value is **not** the average of the values of f at the endpoints, i.e.,

$$\frac{1}{2}[f(0,0) + f(1,1)] = \frac{1}{2}[0 + 1] = \frac{1}{2}.$$

This is due to the nonlinear nature of the function $x^2 + y^2$.

2. If $\rho(x, y, z)$ is the lineal mass/charge density of a thin wire that has the shape of a curve C in $\mathbf{R}^3$, then the total mass/charge of the wire is given by

$$M = \int_C dm = \int_C \rho \, ds. \tag{113}$$

Often, one will see the so-called "differential form" of the above relation,

$$dm = \rho \, ds. \tag{114}$$

This states that the infinitesimal mass dm at a point on the wire is $\rho \, ds$, where ρ is the lineal mass density and ds is the infinitesimal arclength. This can be rearranged to

$$\rho = \frac{dm}{ds}. \tag{115}$$

We have returned to the idea that the lineal density is the limiting ratio of mass to length.

In the case that ρ represents mass density, then the center of mass, $(\bar{x}, \bar{y}, \bar{z})$ of the wire is given by by the following formulas,

$$
\begin{aligned}
\bar{x} &= \frac{1}{M} \int_C x \, dm = \frac{1}{M} \int_C x\rho \, ds, \\
\bar{y} &= \frac{1}{M} \int_C y \, dm = \frac{1}{M} \int_C y\rho \, ds, \\
\bar{z} &= \frac{1}{M} \int_C z \, dm = \frac{1}{M} \int_C z\rho \, ds.
\end{aligned} \tag{116}
$$

Example 1: Let us return to Example 1 done earlier: The integration of the function $f(x, y) = x^2 + y^2$ over the straight line from $(0,0)$ to $(1,1)$. We now consider this straight line to represent a wire of length $\sqrt{2}$. We'll also consider $\rho(x, y) = f(x, y) = x^2 + y^2$ to be the lineal mass density function of the wire. At one end, $\rho(0,0) = 0$ and at the other end $\rho(1,1)$ – as one moves from $(0,0)$ to $(1,1)$, the density of the wire increases. As such, we would expect that the center of mass would not be at the centroid of the wire, i.e., $(1/2, 1/2)$ but closer to $(1,1)$ instead.

Using the parametrization $\mathbf{g}(t) = (x(t), y(t)) = (t, t)$, we computed the total mass of the wire in Example 1:

$$M = \int_C dm = \int_C \rho \, ds = \frac{2\sqrt{2}}{3}. \tag{117}$$

29

It now remains to compute the x and y moments,

$$\int_C x \, dm = \int_C x\rho \, ds. \tag{118}$$

In this case, the integrand evaluated over the curve C will be

$$x(t)\rho(x(t), y(t)) = x(t)[x(t)^2 + y(t)^2] = t(2t^2) = 2t^3. \tag{119}$$

The line integral in (118) becomes

$$\int_C x\rho \, ds = \int_0^1 (2t^3)\sqrt{2} \, dt = 2\sqrt{2} \int_0^1 t^3 \, dt = \frac{\sqrt{2}}{2}. \tag{120}$$

The x-coordinate of the center of mass is then

$$\bar{x} = \frac{\sqrt{2}/2}{2\sqrt{2}/3} = \frac{3}{4}. \tag{121}$$

In a similar fashion, we find that

$$\bar{y} = \bar{x} = \frac{3}{4}. \tag{122}$$

As expected, the center of mass lies closer to the endpoint $(1, 1)$. Moreover, because the wire is straight, the center of mass lies on the wire. (If the wire were bent, this would not be guaranteed.)

Example 2: We now return to Example 3 done earlier: The integration of the same function $f(x, y) = x^2 + y^2$ over the parabolic curve $y = x^2$ from (0,0) to (1,1). We consider this curve to represent a wire. We also consider $\rho(x, y) = f(x, y) = x^2 + y^2$ to be the lineal mass density function of the wire.

Using the parametrization $\mathbf{g}(t) = (x(t), y(t)) = (t, t^2)$, we computed the total mass of the wire in Example 3:

$$M = \int_C dm = \int_C \rho \, ds \approx 0.996. \tag{123}$$

We now compute the x moment,

$$\int_C x \, dm = \int_C x\rho \, ds. \tag{124}$$

The integrand in this integral is

$$x(t)\rho(x(t), y(t)) = x(t)[x(t)^2 + y(t)^2] = t(t^2 + t^4) = t^3 + t^5. \tag{125}$$

The line integral in (124) becomes

$$\int_C x\rho\, ds = \int_0^1 (t^3 + t^5)\sqrt{1 + 4t^2}\, dt \;\;=\;\; \frac{5}{14}\sqrt{5} + \frac{1}{140}$$

$$\approx\;\; 0.805\,. \tag{126}$$

We compute the approximate x-coordinate of the center of mass to be

$$\bar{x} = \frac{1}{M}\int_C x\, dm \approx \frac{0.805}{0.996} \approx 0.808\,. \tag{127}$$

We now compute the y moment,

$$\int_C y\, dm = \int_C y\rho\, ds\,. \tag{128}$$

The integrand in this integral is

$$y(t)\rho(x(t), y(t)) = y(t)[x(t)^2 + y(t)^2] = t^2(t^2 + t^4) = t^4 + t^6\,. \tag{129}$$

The line integral in (128) then becomes

$$\int_C x\rho\, ds = \int_0^1 (t^4 + t^6)\sqrt{1 + 4t^2}\, dt \approx 0.678\,. \tag{130}$$

The y-coordinate of the center of mass is then approximately

$$\bar{y} = \frac{1}{M}\int_C y\, dm \approx \frac{0.678}{0.996} \approx 0.681\,. \tag{131}$$

In summary, the approximate location of the center of mass of this parabolic wire is

$$(\bar{x}, \bar{y}) \approx (0.808, 0.681)\,. \tag{132}$$

We now check if the center of mass lies on the wire (at least to a good approximation):

$$\bar{x}^2 \approx (0.808)^2 \approx 0.652 < \bar{y}\,. \tag{133}$$

This shows that the center of mass $(\bar{x}, \bar{y})$ does not lie on the wire but above the wire, as roughly sketched in the figure below. The location of the center of mass lies closer to – in fact much closer to – the endpoint $(1, 1)$ is due to the nonconstant density function $\rho(x, y) = x^2 + y^2$, which increases significantly as we approach $(1, 1)$.
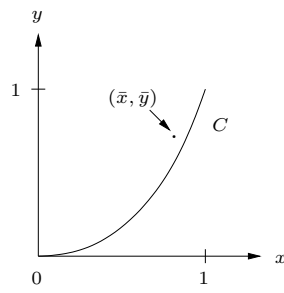
Approximate location of center of mass $(\bar{x}, \bar{y})$ for parabolic wire with density function $\rho(x, y) = x^2 + y^2$.

Lecture 10

Line (Path) integrals of vector-valued functions

(Relevant section from AMATH 231 Course Notes: Chapter 2, Section 2.2)

The integration of a vector field $\mathbf{F}$ over a curve C is a commonly encountered application of vector calculus. It can be used to compute the work done by a nonconstant force acting on a particle that moves along a curve. It can also be used to measure the circulation of a rotating fluid or a magnetic field.

In what follows, we shall motivate our treatment with reference to the computation of work done by a force. As such, it will be useful to recall a fundamental formula from physics. Suppose that a *constant* force $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is applied to a mass m that is constrained to move only in the x-direction. The mass is moved from position $x = a$ to $x = b$. Most of you will quickly answer that the total work done by the force is $W = F_1(b - a)$, which is correct. It is the component of the force $\mathbf{F}$ in the direction of motion that "does the work." This is an example of the more general formula that can be used in $\mathbf{R}^n$:

$$W = \mathbf{F} \cdot \mathbf{d}, \tag{134}$$

where $\mathbf{F}$ is the *constant* force acting on an object and $\mathbf{d}$ is the displacement vector $\overrightarrow{PQ}$ of the mass, where P and Q are, respectively, the initial and final positions of the mass. There is another assumption behind this formula: *that the actual motion of the mass was along the straight line $\overline{PQ}$.*

Now let us consider a more general case: A force, $\mathbf{F}(\mathbf{r})$, which is not necessarily constant in space, is acting on a mass m, as the mass moves along a curve C from point P to point Q as shown in the diagram below.
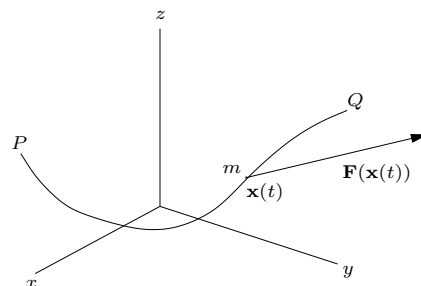
The goal is to compute the total amount of work W done by the force. Clearly Eq. (134) does

not apply here. But the fundamental idea, in the "Spirit of Calculus," is to break up the motion into tiny pieces over which we can use (134) as an approximation. We then "sum up," i.e., integrate, over all contributions to obtain W.

Just as we did for the previous line integration problem, we assume that the curve C can be parametrized, i.e.,

$$\mathbf{x}(t) = \mathbf{g}(t) = (x(t), y(t), z(t)), \qquad t \in [a, b], \tag{135}$$

so that $\mathbf{g}(a)$ is point P and $\mathbf{g}(b)$ is point Q. Now divide the parameter interval $[a, b]$ into n subintervals of length $\Delta t = (b - a)/n$: We do this by defining the partition points $t_k = a + k\Delta t$, $k = 0, 1, 2, \cdots, n$. These points define a set of $n + 1$ points $P_k = \mathbf{g}(t_k) = (x(t_k), y(t_k), z(t_k))$, $k = 0, 1, \cdots, n$ that lie on the curve C, with $P_0 = P$ and $P_n = Q$.

We now come to the major point of this procedure. We shall make two approximations in the computation of the work ΔW_k done by the force $\mathbf{F}$ in moving mass m from point P_{k-1} to point P_k:

1. We shall assume that the force acting on the mass during this time interval is *constant.* From each parameter subinterval $[t_{k-1}, t_k]$, pick a sample point t_k^* so that

$$(x_k^*, y_k^*, z_k^*) = \mathbf{g}(t_k^*) = (x(t_k^*), y(t_k*), z(t_k^*))$$

   is a sample point on the curve extending from P_{k-1} to P_k Now evaluate the force vector $\mathbf{F}$ at this sample point, i.e., $\mathbf{F}(\mathbf{g}(t_k))$. This is the constant force that we shall assume acts on the mass from P_{k-1} to P_k. (For n very large, we don't expect the force $\mathbf{F}$ to change appreciably over this subcurve.)

2. We shall also assume that the motion of the mass is along the straight line segment $\overline{P_{k-1}P_k}$.

These two approximations now allow us to use the "$\mathbf{F} \cdot \mathbf{d}$" formula in (134): the work ΔW_k done by the force over the subcurve $P_{k-1}P_k$ is approximated by

$$\Delta W_k \approx \mathbf{F}(\mathbf{g}(t_k^*)) \cdot \overrightarrow{P_{k-1}P_k}. \tag{136}$$

We still need to express the displacement vector $\overrightarrow{P_{k-1}P_k}$ in terms of the parameter t, in order to be able to produce an integration over t. Note that

$$\begin{aligned}
\overrightarrow{P_{k-1}P_k} &= \mathbf{g}(t_k) - \mathbf{g}(t_{k-1}) \tag{137} \\
&= \frac{\mathbf{g}(t_k) - \mathbf{g}(t_{k-1})}{\Delta t} \Delta t \\
&\approx \mathbf{g}'(t_k^*) \Delta t.
\end{aligned}$$

The final line is another approximation – we have replaced the *mean velocity* over the line segment $\overrightarrow{P_{k-1}P_k}$ with the value of the velocity at the sample point $\mathbf{g}(t_k^*)$. However, as n is increased, the length of the interval $[t_{k-1}, t_k]$ decreases, and the approximation gets better.

Thus the work ΔW_k is now approximated by

$$\Delta W_k \approx \mathbf{F}(\mathbf{g}(t_k^*)) \cdot \mathbf{g}'(t_k^*)\Delta t. \tag{138}$$

The total work is now approximated as a sum over all subcurves:

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k-1}^{n} \mathbf{F}(\mathbf{g}(t_k^*)) \cdot \mathbf{g}'(t_k^*)\Delta t. \tag{139}$$

This is now a Riemann sum involving the *scalar-valued function*

$$f(\mathbf{g}(t)) = \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \tag{140}$$

evaluated at the sample points t_k^*. In the limit $n \to \infty$, this Riemann sum converges to a definite integral:

$$W = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt = \text{``}\int_C \mathbf{F} \cdot d\mathbf{x}.\text{''}. \tag{141}$$

The term

$$d\mathbf{x} = \mathbf{g}'(t)dt \tag{142}$$

represents the *infinitesimal displacement vector* associated with the velocity vector $\mathbf{g}'(t)$. The expression "$\int_C \mathbf{F} \cdot d\mathbf{x}$" denotes the "line integral of the vector field $\mathbf{F}$ over the curve C. But it will be the expression,

$$W = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt, \tag{143}$$

that will be useful for the practical computation of these line integrals.

A comment regarding notation:

Various notations are implied for the path integral of a vector-valued function. You may see the following notations in other books:

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \qquad \int_C \mathbf{F} \cdot d\mathbf{s}. \tag{144}$$

They all mean the same thing – the line integral of the vector field $\mathbf{F}$ over the curve C.

Another interesting note covered in the lecture:

Before going on to some examples, let us perform a slight modification of the above expression for the line integral of a vector field. We shall simply multiply and divide the integrand by the speed $\parallel \mathbf{g}'(t) \parallel$ associated with the parametrization:

$$\int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \frac{\mathbf{g}'(t)}{\parallel \mathbf{g}'(t) \parallel} \parallel \mathbf{g}'(t) \parallel dt \tag{145}$$

But

$$\frac{\mathbf{g}'(t)}{\parallel \mathbf{g}'(t) \parallel} = \hat{\mathbf{T}}(t), \tag{146}$$

The *unit tangent vector* to the curve C at $\mathbf{g}(t)$. And

$$\parallel \mathbf{g}'(t) \parallel dt = ds, \tag{147}$$

the infinitesimal element of arclength along the curve C. The result is that

$$\int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)dt = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \hat{\mathbf{T}}(t) \parallel \mathbf{g}'(t) \parallel dt = \int_C \mathbf{F} \cdot \hat{\mathbf{T}} \, ds. \tag{148}$$

The scalar quantity $\mathbf{F} \cdot \hat{\mathbf{T}}$ is the projection of $\mathbf{F}$ in the direction of $\hat{\mathbf{T}}$, which is precisely the component of $\mathbf{F}$ doing the work in moving the mass in the direction of $\hat{\mathbf{T}}$, which is the direction of the velocity vector $\mathbf{g}'(t)$, the instantaneous direction of motion of the mass at any point on the curve. *This is the proper way to view this integral.*

In a more general setting, the above line integral is adding up the component of $\mathbf{F}$ in the direction of the tangent vector along the curve. We shall return to this important concept later in this chapter.

Some examples:

1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{g}$ where $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ along the curve $\mathbf{g}(t) = (t, t, t)$ for $0 \leq t \leq 1$.

   This parametrization produces a straight line that starts at (0,0,0) and ends at (1,1,1).

   Step 1: Evaluate the velocity vector: $\mathbf{g}'(t) = (1, 1, 1)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Dropping the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for convenience, we simply write $\mathbf{F} = (xyz, y^2, z)$ so that

$$\mathbf{F}(\mathbf{g}(t)) = (x(t)y(t)z(t), y(t)^2, z(t)) = (t^3, t^2, t). \tag{149}$$

Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (t^3, t^2, t) \cdot (1, 1, 1) = t^3 + t^2 + t. \tag{150}$$

We may now evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^1 (t^3 + t^2 + t)dt = \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{13}{12}. \tag{151}$$

2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{g}$ where $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ as in Example 1, but the curve is now $\mathbf{g}(t) = (t, t^2, t^2)$, $0 \le t \le 1$

This parametrization produces a parabolic curve line that also starts at (0,0,0) and ends at (1,1,1).

Step 1: Evaluate the velocity vector: $\mathbf{g}'(t) = (1, 2t, 2t)$.

Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Here,

$$\mathbf{F}(\mathbf{g}(t)) = (x(t)y(t)z(t), y(t)^2, z(t)) = (t^5, t^4, t^2). \tag{152}$$

Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) = (t^5, t^4, t^2) \cdot (1, 2t, 2t) = t^5 + 2t^5 + 2t^3 = 3t^5 + 2t^3. \tag{153}$$

We may now evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^1 (3t^5 + 2t^3)dt = \frac{1}{2} + \frac{1}{2} = 1. \tag{154}$$

Note that this result differs from that of Example 1. The line integrals had the same endpoints but different paths. There is no guarantee that the results will be the same.

An important application in Physics

Proposition: A mass m moves in $\mathbf{R}^3$ under the influence of a force $\mathbf{F}$ according to Newton's Law $\mathbf{F} = m\mathbf{a}$. The mass moves from A to point B along a trajectory $\mathbf{g}(t)$ which we shall denote as curve C_{AB}. Then the work W done by the force $\mathbf{F}$ along C_{AB} is equal to the change in kinetic energy of the mass, i.e.

$$W = K(B) - K(A) = \Delta K, \tag{155}$$

where K denotes the kinetic energy of the particle.

Proof: The total work done by the force is

$$W = \int_{C_{AB}} \mathbf{F} \cdot d\mathbf{x} \tag{156}$$
$$= \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) \, dt,$$

where a and b denote the times that the particle is at A and B, respectively. But $\mathbf{g}'(t) = \mathbf{v}(t)$, the velocity of the mass. At all points on the trajectory, Newton's Law is obeyed, implying that

$$\mathbf{F}(\mathbf{g}(t)) = m\mathbf{a}(t) = m\mathbf{v}'(t). \tag{157}$$

We substitute this result into the work integral:

$$W = \int_a^b m\mathbf{v}'(t) \cdot \mathbf{v}(t) \, dt. \tag{158}$$

But recall that

$$\frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}'(t) \cdot \mathbf{v}(t). \tag{159}$$

or

$$\mathbf{v}'(t) \cdot \mathbf{v}(t) = \frac{1}{2}\frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t) = \frac{1}{2}\frac{d}{dt} \parallel \mathbf{v}(t) \parallel^2 . \tag{160}$$

Therefore

$$W = \frac{1}{2}m \int_a^b \frac{d}{dt} \parallel \mathbf{v}(t) \parallel^2 \, dt \tag{161}$$
$$= \frac{1}{2}m \parallel \mathbf{v}(b) \parallel^2 - \frac{1}{2}m \parallel \mathbf{v}(a) \parallel^2$$
$$= K(B) - K(A)$$
$$= \Delta K.$$

You probably saw this result for one-dimensional motion in your first-year Physics course. Note, however, that we have **not** made any assumptions on $\mathbf{F}$ – it does not have to be conservative. For example, it also holds for frictional forces, which are nonconservative. The proof of this result for *conservative* forces is much simpler, since we can use the fact that total mechanical energy is conserved. But we must use the Generalized Fundamental Theorem of Calculus, the next topic.