University of Waterloo

Faculty of Mathematics

Department of Applied Mathematics

AMATH 231: Calculus 4

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Lecture Notes

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Lecture 1

Introduction

As written in the Preface to the AMATH 231 Course Notes (by J. Wainwright and G. Tenti):

> This course has two main themes:
>
> 1. the calculus of vector fields, and
> 2. Fourier analysis.
>
> The first theme, which includes the famous integral theorems associated with the names of Green, Gauss and Stokes, represents the culmination of the traditional Calculus sequence. This material provides the mathematical foundation for continuum mechanics (AMATH 361), fluid dynamics (AMATH 463) and electromagnetic theory (PHYS 252 & 253) and is of importance for partial differential equations (AMATH 353).
>
> The second theme, Fourier analysis, is built on the remarkable idea that a variety of complicated functions can be synthesized from pure sine and cosine functions. This material provides the mathematical foundation for signal and image processing (course under development)* and is of importance for PDEs (AMATH 353) and quantum mechanics (AMATH 373).

*The course was under development when these notes were first written. Since that time, the course, AMATH 391, "From Fourier to Wavelets," has been offered several times by this instructor.

A few more remarks about these two major themes:

Vector Calculus

Historically, **vector calculus**, the calculus of vector fields, was originally developed in order to understand a number of physical phenomena occurring in nature, e.g., fluid flow, electric and magnetic fields, gravitation. Note that these phenomena require the concept of a *field* which exists in space and which can be used to explain their behaviour.

The vector fields relevant to these natural phenomena are most often represented by vector-valued functions, i.e.,

$$\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3 . \tag{1}$$

- The **input** to such functions is a point $\mathbf{x}$ in three-dimensional space with coordinates that may be denoted as (x, y, z) or (x_1, x_2, x_3), i.e.,

$$\mathbf{x} = (x, y, z) \qquad \text{or} \qquad \mathbf{x} = (x_1, x_2, x_3) . \tag{2}$$

  (Sometimes, it is more convenient to use the second notation.)

- The **output** of such vector-valued functions is also an ordered triple, the components of which could be functions of all physical coordinates, i.e.,

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(x, y, z) \\ &= (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) . \end{aligned} \tag{3}$$

(You don't have to get worried about any details right now. All of these concepts will be explained in more detail a little later.)

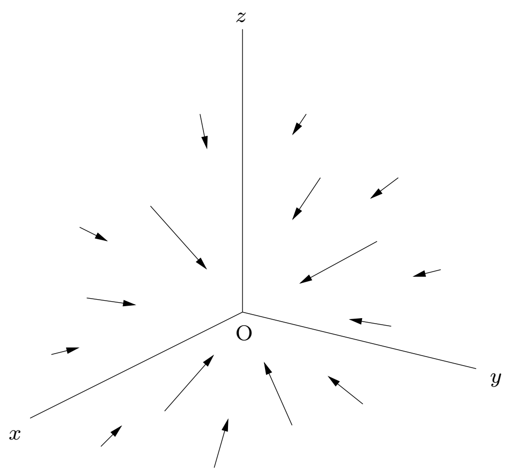
Because of its application to natural phenomena, it is often convenient to pictorially represent the output,

$$(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) , \tag{4}$$

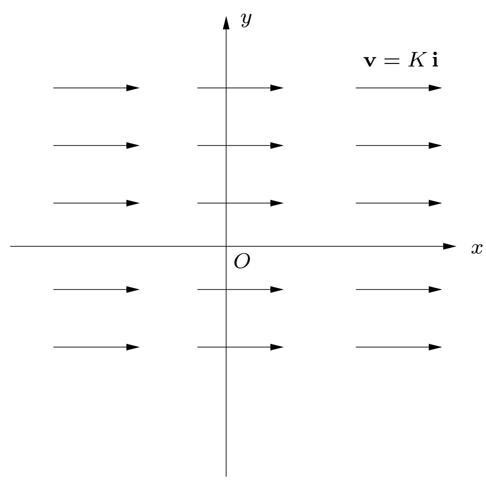
as a **vector** in $\mathbb{R}^3$, i.e., an arrow which emanates from the point (x, y, z) in the direction dictated by the ordered triple $(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$. Technically, this vector could be expressed mathematically as

$$\mathbf{v}(x, y, z) = f_1(x, y, z)\,\mathbf{i} + f_2(x, y, z)\,\mathbf{j} + f_3(x, y, z)\,\mathbf{k} , \tag{5}$$

where **i**, **j** and **k** denote the unit vectors which point in the positive x, positive y and positive z directions in $\mathbb{R}^3$, respectively. Some examples of vector fields relevant to physics are sketched below.



The gravitational force field vector $\mathbf{F}(\mathbf{x}) = -\dfrac{GMm}{\|x\|^3}\mathbf{x}$.



Vector field representing uniform fluid flow in the plane – all fluid particles are travelling at constant velocity $\mathbf{v} = K\,\mathbf{i}$.

We then study the differential and integral calculus associated with these vector-valued functions. To do so, we'll take the ideas from first-year calculus, formulated in one-dimension and extend them to higher dimensions. But you've already done this, to some extent, in MATH 237 (or equivalent), where you studied **scalar-valued functions of several variables**, i.e., functions of the form,

$$f : \mathbb{R}^n \to \mathbb{R}\,. \tag{6}$$

So things shouldn't be that bad!

Among the other things that we shall be developing in this section of the course are:

- The idea of **path integrals of vector fields**. In your first-year Physics and Calculus courses, you most probably considered the situation of a nonconstant force $F(x)$ acting on a mass m, moving it along a straight line – the x axis – from a point $x = a$ to $x = b$. The total work W done by the force on the mass m is the definite integral,

$$W = \int_a^b F(x)\,dx\,. \tag{7}$$

  In this course, we shall compute more general situation in $\mathbb{R}^3$: the work W done by a nonconstant force $\mathbf{F}(\mathbf{x})$ acting on a mass m over a **curve** $C \subset \mathbb{R}^3$. (After all, the earth does not move in a straight line around the sun!) The work W is given by the so-called **path integral**

$$W = \int_C \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}\,. \tag{8}$$

- We shall be using the fundamental theorems of vector calculus (e.g., Gauss' Divergence Theorem) to derive **conservation laws** which yield the following important equations (more correctly, partial differential equations) in applied mathematics, physics and engineering:

  - The **heat or diffusion equation**,
  - **Maxwell's equations for electromagnetism**.

Finally, we'll also mention show very briefly in this course that vector-valued functions are also very useful in a variety of other applications that have nothing to do with the physical applications listed above. Here, we'll simply state that in image processing, vector valued functions are a natural representation of the following:

1. simple colour images (i.e., "RGB images"),
2. "hyperspectral images" obtained from remote sensing of the earth's surface
3. diffusion magnetic resonance images (dMRI).

Fourier analysis: Fourier series and Fourier transforms

Historically, the idea of Fourier analysis goes back to the pioneering work of Jean Joseph Fourier (1768-1830) during the very early 1800's (i.e., 1802-1810) in which he sought to understand the behaviour of heat, i.e., heat propagation in solides, heat loss by radiation and heat conservation. Fourier was a pioneer in the true sense of the word – he worked with ideas and concepts that were not yet properly formulated (at least in the pure mathematical sense). His idea was to use series expansions of functions to solve the so-called **heat equation** – which he formulated, on the basis of physical arguments.

Recall the idea of the **power series expansion** of a real-valued function of a single real variable, say, $f(x)$ about a point $x_0 \in R$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \,, \quad |x - x_0| < R \,. \tag{9}$$

Here, $R \geq 0$ is the **radius of convergence** of the series. (You'll also recall that the coefficients a_n can be computed from the derivatives $f^n(x_0)$, provided that they exist.)

Fourier had the idea of expressing a function $f(x)$ as a series of sine and cosine functions, i.e., replacing the functions $g_n = (x - x_0)^n$ in Eq. (9) with appropriate sine and cosine functions. Such expansions are usually written in the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\, a_n \cos(nx) + b_n \sin(nx) \,] \,, \quad x \in (-\pi, \pi) \,. \tag{10}$$

(The reason for the term $\frac{1}{2}$ multiplying the coefficient a_0 will be explained later.) Note that since all the sine and cosine terms are 2π-periodic, the function $f(x)$ is 2π-periodic, i.e.,

$$f(x + 2\pi) = f(x) \,. \tag{11}$$

This explains the "$x \in (-\pi, \pi)$" at the end of the equation. We'll also explain later why the endpoints π and $-\pi$ are omitted.

The expansion in Eq. (10) in terms of functions may seem somewhat overwhelming - but it shouldn't. Or at least it won't after we spend a few lectures on the subject. This expansion can be viewed as a generalization of the well-known expansion of an element $\mathbf{x} \in \mathbb{R}^n$ in terms of a set of basis vectors $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_n\}$, i.e.,

$$\mathbf{x} = \sum_{k=1}^{n} c_k \mathbf{v}_k \,, \tag{12}$$

And in the special case where the $\mathbf{v}_k$ form an **orthogonal basis**, and perhaps even an **orthonormal basis**, the coefficients c_k are easily computed in terms of inner products involving the vector $\mathbf{x}$ and the $\mathbf{v}_k$.

Returning to the expansion in (10), the (infinite) set of functions,

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots\}\,, \tag{13}$$

form a **complete, orthogonal basis set** for an appropriate **vector space of functions** $f : (-\pi, \pi) \to \mathbb{R}$, which we shall simply denote as $\mathcal{F}$. The vector space of functions, $\mathcal{F}$, is **infinite-dimensional**.

What are the two major differences between the expansion (12) in $\mathbb{R}^n$ (finite-dimensional space) to the Fourier series expansion (10) in the (infinite-dimensional) space $\mathcal{F}$?

1. The upper index n is now ∞.

2. Each (constant) vector $\mathbf{v}_k$ is now a function of x.

Finally, we'll mention that the space $\mathcal{F}$ is well known in a number of applications ranging from signal processing to quantum mechanics. It is the function space commonly denoted as $\mathcal{L}^2(-\pi, \pi)$, the space of **square-integrable** functions on $(-\pi, \pi)$. It is also a **Hilbert space** (a complete, inner product space). But more on this later.

Perhaps it will serve as a motivation to know that the method of Fourier series provides the foundations for the so-called **JPEG** image compression method with which you have all undoubtedly been in contact. Whenever you download an image with the suffix **.jpg**, you are downloading an image which has been compressed with the JPEG standard. You are actually not downloading an image, but instead some kind of compressed version of expansion coefficients analogous to the a_n in Eq. (10)!