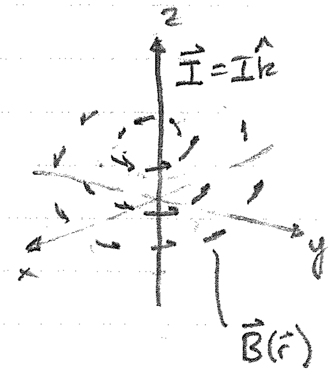
The main purpose of this lecture was to illustrate the effectiveness of Vector Calculus to model basic electromagnetic phenomena.

Biot-Savart Effect for Continuous Charge Distributions

Recall briefly the main idea of the Biot-Savart effect: A current $\vec{I}$ of electric charge in the direction of unit vector $\hat{u}$, ie $\vec{I} = I\hat{u}$ (thin wire aligned in direction of $\hat{u}$) produces a magnetic field $\vec{B}(\vec{r})$ according to



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi} \frac{\hat{u} \times \vec{r}}{\|\hat{u} \times \vec{r}\|^3} \qquad (1)$$

In the case that $\hat{u} = \hat{k}$, ie the wire lies along the z-axis, the field $\vec{B}(\vec{r})$ is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi} \left[-\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath} \right] \qquad (2)$$

A most interesting property of this field is that

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \vec{0} \quad \text{for } \vec{r} \text{ not on the wire, ie } (x,y) \neq (0,0)$$

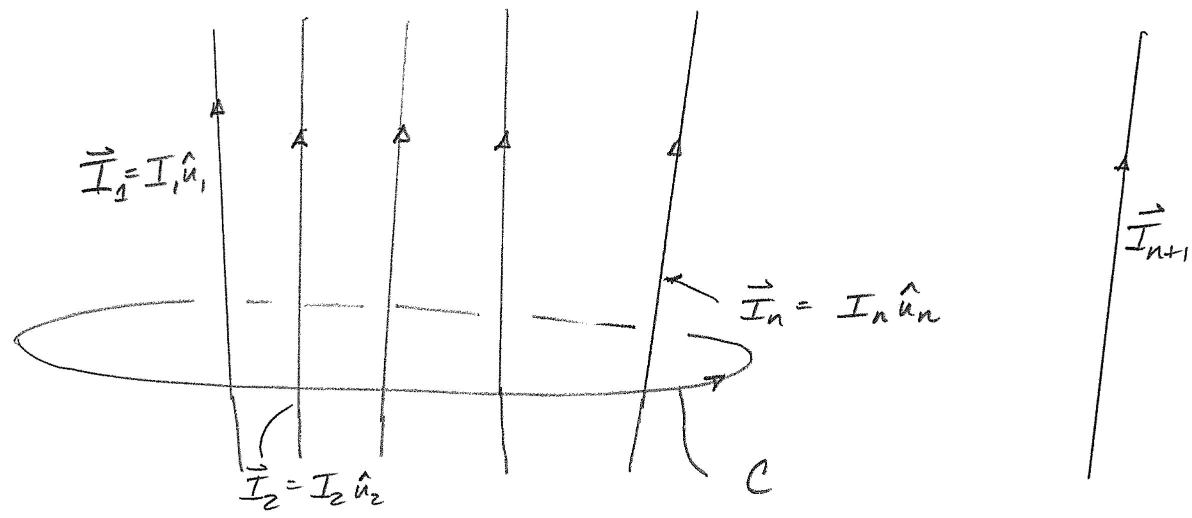
For any simple closed C' curve that encloses (but does not intersect) the wire (the z-axis in the case of (2)), the circulation of $\vec{B}(\vec{r})$ over this curve is

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I. \qquad (3)$$

(We computed this circulation for the special case that C is a circle of radius R centered at the wire and lying perpendicular to it. In the current Problem Set (No. 7), you are asked to show that the circulation is the same for all curves C enclosing the wire. The proof involves a rather straightforward use of Stokes' Theorem.)

The "thin wire" supporting the current in the above model is analogous to the "point charge" of electrostatic charge. This is reflected in the singularity of $\vec{B}(\vec{r})$ for $\vec{r} \in$ wire (z-axis in (2)). Very shortly, we shall "smear" out the current source – as we did for electric charge – via a continuous charge density function $\rho(\vec{r})$

But before we do this, we mention that one can proceed in a manner analogous to what was done for Gauss' Law and electric charges. We can consider a finite collection of wires that are not necessarily parallel, but which we assume, for simplicity, do not intersect.



Each of these current-carrying wires with current $\vec{I}_k$ will produce a magnetic field $\vec{B}_k(\vec{r})$ according to Eq. (1). The net magnetic field $\vec{B}(\vec{r})$ is the vector sum of these fields, i.e

$$\vec{B}(\vec{r}) = \sum_{k=1}^{n} \vec{B}_k(\vec{r})$$

$$= \frac{\mu_0}{2\pi} \sum_{k=1}^{n} I_k \frac{\hat{u}_k \times \vec{r}}{\|\hat{u}_k \times \vec{r}\|^3} \qquad (4)$$

For any simple closed curve C that encloses all wires, as shown above (the wires are assumed to have infinite length) the circulation of $\vec{B}(\vec{r})$ is

$$\oint_C \vec{B} \cdot d\vec{r} = \sum_k \oint_C \vec{B}_k \cdot d\vec{r} = \mu_0 I, \quad \text{where} \quad I = \sum_{k=1}^{n} I_k \qquad (5)$$

This result is once again based on Stokes' Theorem. For any wire $\vec{I}_{n+1}$ not enclosed by the wire, its contribution to the circulation integral is zero, once again by Stokes' Theorem, since $\vec{\nabla} \times \vec{B}_{n+1}(\vec{r}) = 0$ for $\vec{r} \notin$ wire $\vec{I}_{n+1}$

CONTINUOUS DISTRIBUTION OF CURRENT

We view electric current as we would a fluid moving in space. Let $\rho(\vec{r},t)$ denote the charge density function (charge / unit volume at $\vec{r}$ at time t) for all $\vec{r} \in D$, $D \subset \mathbb{R}^3$. Let $\vec{v}(\vec{r},t)$ denote the velocity of this fluid of electric charge at $\vec{r}$, time t. The current density vector (current = charge × velocity) is defined as

$$\left.\begin{aligned}[\rho] &= \frac{\text{charge}}{\text{volume}} = \frac{q}{L^3} \\ [v] &= \frac{\text{length}}{\text{time}} = \frac{L}{T}\end{aligned}\right\} [J] = \frac{q}{L^2 T}$$

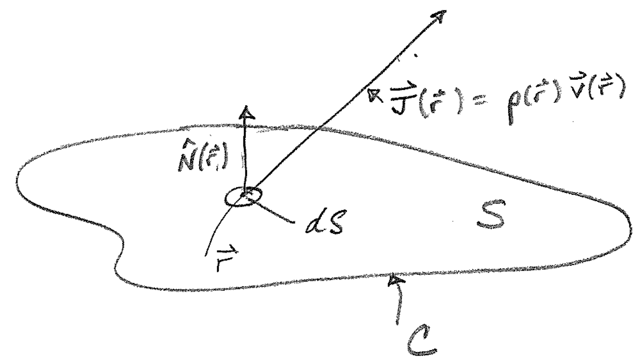
$$\vec{J}(\vec{r},t) = \rho(\vec{r},t)\,\vec{v}(\vec{r},t) \qquad \vec{r} \in D \qquad (6)$$

In what follows, we assume that the flow is an equilibrium flow, i.e. it doesn't change in time, so that the above equation becomes

$$J(\vec{r}) = \rho(\vec{r})\,\vec{v}(\vec{r}) \qquad \vec{r} \in D \qquad (7)$$

(If $\vec{J}$ changes in time, then the situation changes, and we have to consider the change in the electrostatic field $\vec{E}(\vec{r},t)$.)

Now let $S \subset D$ be a C^1, orientable surface with boundary



$\partial S = C$, a closed, simple C^1 curve.

Let $\hat{N}(\vec{r})$ denote the unit normal to S at $\vec{r} \in S$.

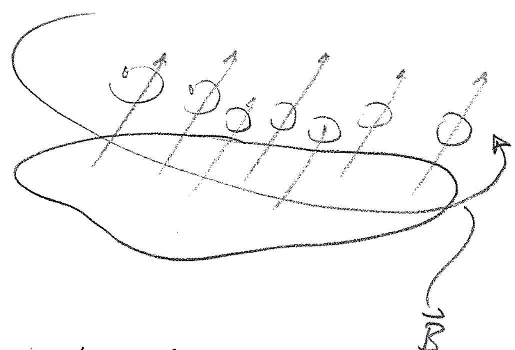
The element of current $dI(\vec{r})$ crossing the surface area element dS centered at $\vec{r} \in S$ is given by

$[dI] = [J][dS] = \frac{q}{L^2 T} \cdot L^2$

$$dI(\vec{r}) = J(\vec{r}) \cdot \hat{N}(\vec{r})\, dS \,. \qquad (8) \qquad = \frac{q}{T} \checkmark$$

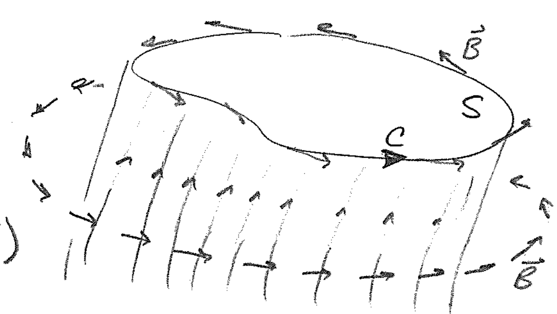
The total current crossing surface S is obtained by integrating over S:

$$I = \text{"}\iint_S dI\text{"} = \iint_S \vec{J} \cdot \hat{N}\, dS \qquad (9)$$



Each element of current $dJ(\vec{r})$ will produce an element of magnetic field that contributes to the total magnetic field $\vec{B}(\vec{r})$ produced by the current flowing through surface S

We can think of surface S as being the surface produced by cutting through a thick wire through which the electric current flows, as defined by $\vec{J}(\vec{r})$



The circulation of the net magnetic field vector $B(\vec{r})$ over the boundary curve C will be the total contribution of all elements of current $dJ(\vec{r})$ on the surface S:

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 \iint_S \vec{J} \cdot \hat{N}\, dS \qquad (10)$$

line integral $\qquad$ surface integral

This is the continuous, "spread out" version of the collection of wires result in Eq. (5).

Assuming that $\vec{B}$ and its derivatives are continuous on S, we apply Stokes' Theorem to the LHS of (10):

$$\iint_S (\vec{\nabla} \times \vec{B}) \cdot \hat{N}\, dS = \mu_0 \iint_S \vec{J} \cdot \hat{N}\, dS \qquad (11)$$

Rearranging:

$$\iint_S \left[\vec{\nabla} \times \vec{B} - \mu_0 \vec{J} \right] \cdot \hat{N}\, dS = 0 \qquad (12)$$

We now rely on the assumptions that

1) the integrand is continuous

2) the surface S is arbitrary

to conclude that

$$\boxed{\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 J(\vec{r}) \qquad \vec{r} \in D \qquad (13)}$$

This is "Ampère's Circuit Law:"

"Where there is current, the magnetic field has circulation."

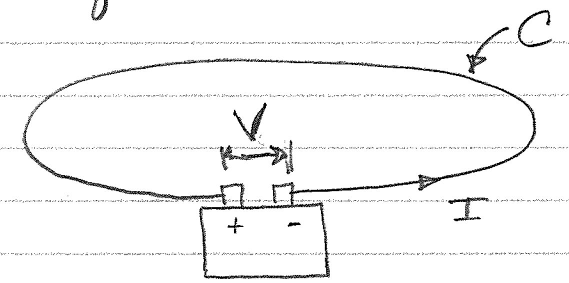
(Compare this to Maxwell's First Law, which we derived earlier, $\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\varepsilon_0} \rho(\vec{r})$: Where there is charge, there is divergence of the electrostatic field.)

We'll return to this equation very shortly.

FARADAY'S LAW

A few "elementary" facts and ideas about circuits, magnetic fields and "induction" are in order.

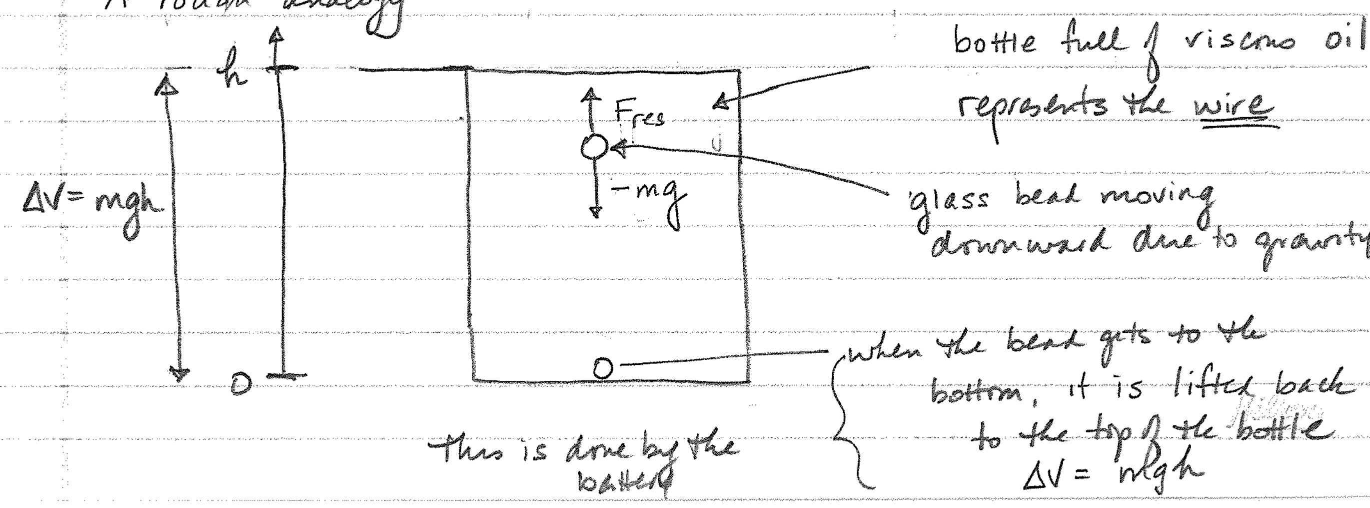
Schematic of a "circuit" with a battery



the "voltage drop V" across the poles of the battery produces a current I along a wire, represented by curve C

The current is produced by electrons moving in the wire (classical picture). There is resistance to this motion, which means that a force $\vec{F}(\vec{x})$ is needed at each point $\vec{x}$ of the wire to "push" the electrons along

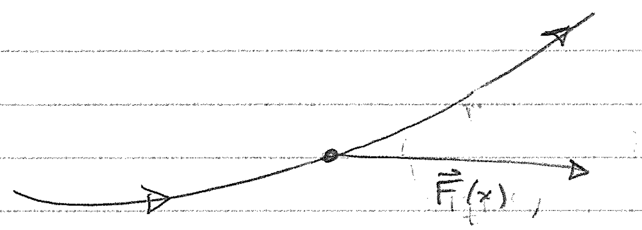
A rough analogy



The battery may be viewed as the source of

1) the gravitational force $-mg$ that moves the bead through the oil

2) the source of a potential energy "boost" that lifts the bead from the bottom to the top of the bottle ...



At each point $\vec{x} \in C$, there is a force $\vec{F}(\vec{x})$ that acts on each charge to "push" it through the wire. (of course, it is the tangential component of this force)

If we let $\vec{E}(\vec{x})$ be the force per unit charge, then the total work done by this force over the circuit/curve C is

$$W = \oint_C \vec{E} \cdot d\vec{x} = -\underbrace{\text{"electromotive force"}}_{\mathcal{E}} \text{ (EMF)}$$

This work is equal and opposite to the voltage drop V across the battery, i.e.

$$\mathcal{E} = V = -\oint_C \vec{E} \cdot d\vec{x}$$

We use a "-" sign since

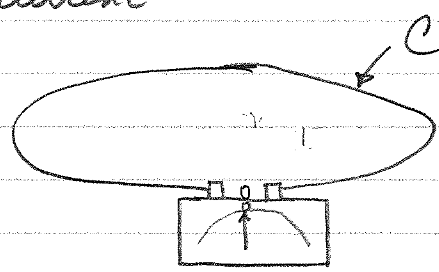
This implies that

$$\oint \vec{E} \cdot d\vec{x} + V = 0$$

Kirchoff's Law: the sum of voltage drops around a circuit is zero

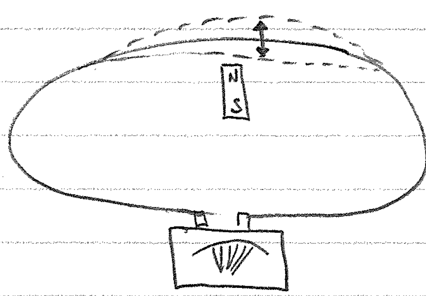
INDUCTION

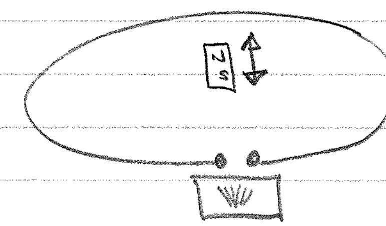
Let's now consider a closed circuit with no battery, implying no current

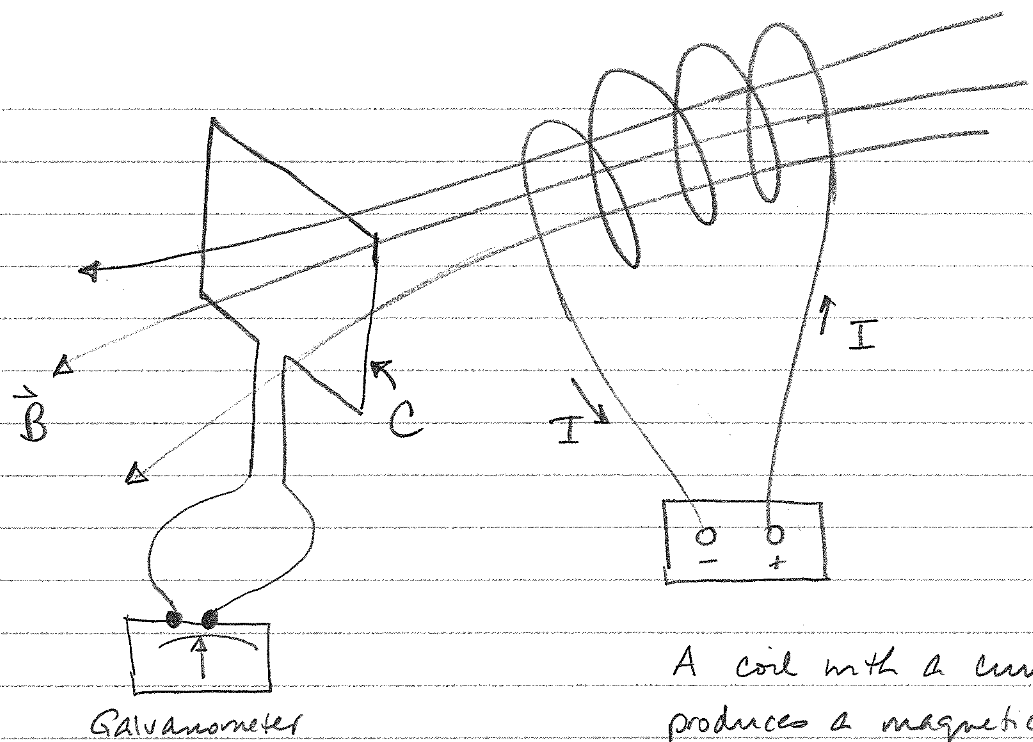


Galvanometer
which detects current

If we move the wire through a magnetic field, a current is produced. Or if we move the magnet, ie change the magnetic field, we produce a current:







A coil with a current produces a magnetic field B

If we either
i) move the coil
or
ii) change the current I
or
iii) both
we change the magnetic field $\vec{B}$
and produce a current in circuit C (which could also be a coil)

Note: If $I = K$ (constant) in the coil, there is no current in C

The current produced in the circuit C must be due to an "electromotive force" $\vec{E}(\vec{x})$ (vector) which acts in the same way that a battery would, if it were connected to wire C.

FARADAY'S LAW:

$$\text{The} \left\{ \begin{array}{l} \text{voltage change} \\ \text{around a loop} \\ \text{of infinitesimally} \\ \text{thin wire} \end{array} \right\} = \left\{ \begin{array}{l} \text{electromotive} \\ \text{force} \quad \oint_C \vec{E} \cdot d\vec{x} \end{array} \right\}$$

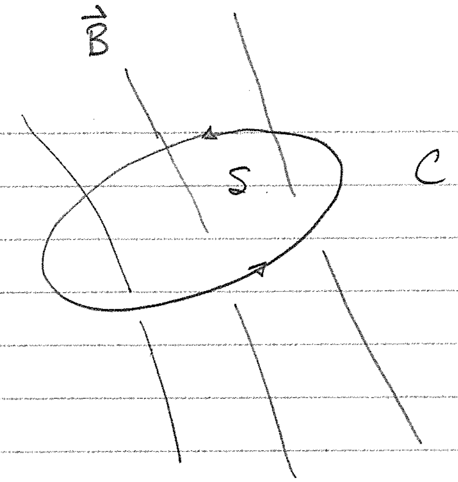
is proportional to the <u>negative</u> of the time rate of change of the magnetic flux of the loop.

<u>Comment:</u> Why "negative"

A: Because of Lenz' Law. Nature always reacts in a way to try to oppose the change being imposed on it.

The change in electromotive force, i.e change in voltage drop produces a current that, in turn, produces a magnetic flux that opposes the change in magnetic flux applied to the circuit.

Circuit C:



Voltage drop $\quad \oint_C \vec{E}\cdot d\vec{x}$

Flux through the loop: Consider a fixed surface S with C as boundary curve. The flux of magnetic field $\vec{B}$ through C is

$$\iint_S \vec{B}\cdot\hat{N}\,dS$$

Notes: 1. In the AMATH 231 Course Notes, "$\vec{H}$" is used to denote the magnetic field. Here, we use $\vec{B}$. In standard notation, $\vec{H}$ is used to denote something else.)

2. Can we use any surface S with boundary C?

The answer is "Yes", which is a consequence of the fact that $\vec{\nabla}\cdot\vec{B}=0$. We'll return to this point later.

Faraday's law may now be expressed mathematically as follows:

$$\oint_C \vec{E} \cdot d\vec{x} \quad \propto \quad -\frac{d}{dt}\left[\iint_S \vec{B} \cdot \hat{N} \, dS\right]$$

which is often written as

$$\frac{d}{dt}\iint_S \vec{B} \cdot \hat{N} \, dS = -C\int_C \vec{E} \cdot d\vec{x} \qquad C>0$$

Since S is time independent, LHS becomes

$$\underbrace{\iint_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{N} \, dS}_{\text{surface integral}} = -C \underbrace{\iint_C \vec{E} \cdot d\vec{x}}_{\text{line integral (circulation}}$$

$$= -C\iint_S (\vec{\nabla} \times \vec{E}) \cdot \hat{N} \, dS \quad \text{(Stokes' Theorem)}$$

$$\iint_S \left[\frac{\partial B}{\partial t} + C\vec{\nabla} \times \vec{E}\right] dS = 0$$

Assumptions

1) Integrand is continuous

2) Surface S is arbitrary (!! But $\partial S = C$ in all cases!)

$$\Rightarrow \quad \frac{\partial \vec{B}}{\partial t} + C\,\vec{\nabla} \times \vec{E} = 0$$

In fact, in SI units, $C = 1$

$$\boxed{\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0}$$