Lecture 35

Fourier series - the sequel

The last couple of lectures, which dealt with complex Fourier series and the idea of the "Fourier spectrum," gave a small glimpse of how Fourier series are useful in the analysis of music. The oscillatory and periodic nature of the sine and cosine functions make them particularly suited for musical sounds.

As has been mentioned on several occasions during this course, Fourier series are a particular example of the expansion of functions in terms of an orthogonal basis set of functions. You may have encountered other examples of orthogonal or orthonormal basis sets of functions, for example, the set of **Legendre polynomials** over the interval $[-1, 1]$. This basis set is useful in many areas of applied mathematics and theoretical/mathematical physics. It provides the basis of the so-called **spherical harmonics** which are employed in solutions of Laplace's equation in three dimensions, as well as Schrödinger's equation for the hydrogen atom.

Fourier series are also important in the study of **partial differential equations**, as you will see if you take a course such as AMATH 353, "Partial Differential Equations I." (Recall that J. Fourier was the originator of this idea.) In what follows, we provide a very brief glimpse of this idea – a small preview of things to be seen in AMATH 353.

Let us consider the one-dimensional heat equation, derived earlier in the course (Week 9, Lecture 21) using conservation laws and the Divergence Theorem. We shall use it to determine the time evolution of the temperature field over a thin steel rod of constant cross-sectional area located on the x-axis in the interval $[-\pi, \pi]$. In what follows, the temperature of the rod at a point $x \in [-\pi, \pi]$ and time t will be denoted as $T(x, t)$. We also suppose that at time $t = 0$, the temperature at any point of the rod is given by

$$T(x, 0) = f(x)\,, \qquad -\pi \leq x \leq \pi\,, \tag{1}$$

where $f(x)$ is some prescribed function.

The time evolution of $T(x, t)$ is determined by the heat equation,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}\,, \tag{2}$$

where κ, the thermal diffusivity of the rod, is assumed to be constant (homogeneous rod).

The first thing to note, as Fourier did, is that the function $f(x)$ in Eq. (1) may be represented as a Fourier seris of the form,

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + \sum_{n=1}^{\infty} B_n \sin nx\,, \tag{3}$$

where the A_n and B_n coefficients may be computed using the well-known formulas derived in the first lecture of the section on Fourier series. (We use capital letters here for reasons that will become clear below.)

The temperature function $T(x,t)$ may be viewed as a **time-varying** function on $[-\pi,\pi]$. At any time t, this function could be represented as a Fourier series of the form given in Eq. (3). With this in mind, we shall let $T(x,t)$ be represented by a **Fourier series with coefficients that vary in time**, i.e., we assume an expansion of the form

$$T(x,t) = \frac{1}{2}a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos nx + \sum_{n=1}^{\infty} b_n(t) \sin nx\,, \tag{4}$$

with the condition that at time $t=0$, the coefficients $a_n(t)$ and $b_n(t)$ coincide with the expansion coefficients of $f(x)$, i.e.,

$$a_n(0) = A_n\,,\ n \geq 0\,, \qquad b_n(0) = B_n\,,\ n \geq 1\,. \tag{5}$$

Note: Time-dependent series expansions such as Eq. (4), in which the coefficients are time-varying, are employed in many areas of applied mathematics and theoretical/mathematical physics, e.g., vibration problems, electromagnetism and quantum mechanics.

The next step is to substitute the assumed expansion in (4) into the heat equation (2). We'll first compute the required derivatives, assuming that termwise differentiation with respect to both the t and x variables is permitted. Firstly,

$$\frac{\partial T}{\partial t} = \frac{1}{2}a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) \cos nx + \sum_{n=1}^{\infty} b_n'(t) \sin nx\,, \tag{6}$$

where the primes indicate differentiation with respect to time t.

We now compute the necessary spatial partial derivatives,

$$\frac{\partial T}{\partial x} = -\sum_{n=1}^{\infty} n\, a_n(t) \sin nx + \sum_{n=1}^{\infty} n\, b_n(t) \cos nx\,, \tag{7}$$

and

$$\frac{\partial^2 T}{\partial x^2} = -\sum_{n=1}^{\infty} n^2\, a_n(t)\cos nx - \sum_{n=1}^{\infty} n^2\, b_n(t)\sin nx\,. \tag{8}$$

Now substitute Eqs. (6) and (8) into the heat equation (2),

$$\frac{1}{2}a_0'(t) + \sum_{n=1}^{\infty} a_n'(t)\cos nx + \sum_{n=1}^{\infty} b_n'(t)\sin nx = -\kappa\sum_{n=1}^{\infty} n^2\, a_n(t)\cos nx - \kappa\sum_{n=1}^{\infty} n^2\, b_n(t)\sin nx\,. \tag{9}$$

Collection of like terms in $\cos nx$ and $\sin nx$ yields the following equation,

$$\frac{1}{2}a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t) + \kappa n^2 a_n(t)\right]\cos nx + \sum_{n=1}^{\infty}\left[b_n'(t) + \kappa n^2 b_n(t)\right]\sin nx = 0\,. \tag{10}$$

This equation has to be satisfied for all $t \geq 0$ and for all $x \in [-\pi, \pi]$. Because of the linear independence of the $\cos nx$ and $\sin nx$ functions over $[-\pi, \pi]$ – which includes the function $\cos 0x = 1$ – the first term of the above equation along with all terms in square brackets must vanish for all t, i.e.,

$$\frac{1}{2}a_0'(t) = 0\,, \qquad \Longrightarrow \qquad a_0(t) = C = A_0\,, \tag{11}$$

and

$$\begin{aligned} a_n'(t) + \kappa n^2 a_n(t) &= 0\,, \\ b_n'(t) + \kappa n^2 b_n(t) &= 0\,, \quad n \geq 1\,. \end{aligned} \tag{12}$$

These are simple first-order differential equations for the functions $a_n(t)$ and $b_n(t)$ with solutions,

$$a_n(t) = A_n e^{-\kappa n^2 t}\,, \qquad b_n(t) = B_n e^{-\kappa n^2 t}\,, \tag{13}$$

where we have imposed the initial conditions in Eq. (5). The net result is that the temperature function $T(x,t)$ which is the solution to the heat equation (2) that satisfies the initial condition in (1) is given by

$$T(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\kappa n^2 t}\cos nx + \sum_{n=1}^{\infty} B_n e^{-\kappa n^2 t}\sin nx\,. \tag{14}$$

In the above treatment, we have ignored the role of **boundary conditions** – for example, (i) whether or not heat is allowed to escape from the rod through one or both ends or (ii) whether one or both ends of the rod are kept at constant temperatures or (iii) a combination of (i) and (ii). These are additional complications that you will study in AMATH 353. For the moment, we simply notice that the time evolution of the coefficients $a_n(t)$ and $b_n(t)$ as given in Eqs. (11) and (13) is rather trivial: $a_0(t)$ is constant and

$$a_n(t) \to 0\,, \quad b_n(t) \to 0\,, \quad \text{as} \quad t \to \infty \quad \text{for} \quad n \geq 1\,. \tag{15}$$

Indeed, from Eq. (14), we see that for all $x \in (-\pi, \pi)$, to $f_p(x)$),

$$T(x,t) \to \frac{1}{2}A_0\,, \quad \text{as} \quad t \to \infty\,. \tag{16}$$

(Note that we are not including the endpoints for the same reasons discussed earlier in the course, i.e., pointwise convergence of the 2π-periodic function represented by the Fourier series.) Let us now recall what the coefficient A_0 actually is:

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,. \tag{17}$$

From (16), this implies that

$$T(x,t) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx\,. \tag{18}$$

The right hand side of this equation is **average value of the initial temperature function over the rod**. This indicates that the total amount of heat energy in the rod has been conserved. It has simply been redistributed from regions of higher temperature to regions of lower temperature until a constant temperature over the rod is achieved.