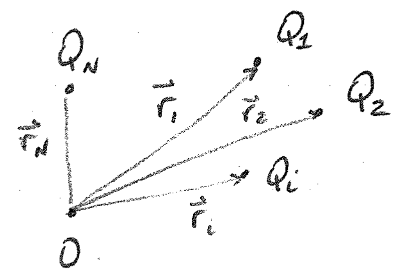
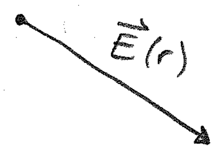
# Surface Integrals / Divergence Theorem: Application to Continuous Charge Distributions

In the last lecture, we considered a finite collection of point charges $Q_i$ situated at positions $\vec{r}_i$, $1 \le i \le N$




Given an "observation point" $\vec{r} \in \mathbb{R}^3$, each of these charges will contribute to a total electrostatic field vector $E(\vec{r})$

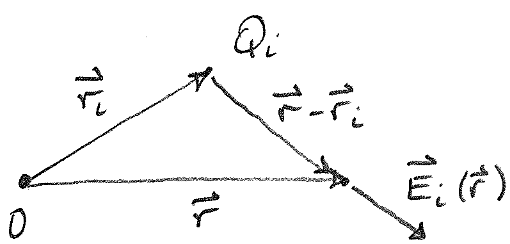



$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \vec{E}_i(\vec{r}) \tag{1}$$

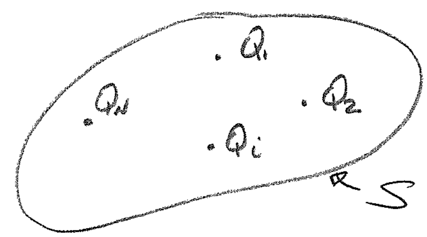
where

$$\vec{E}_i(\vec{r}) = \frac{Q_i}{4\pi\varepsilon_0} \frac{1}{\| \vec{r} - \vec{r}_i \|^3} \left[ \vec{r} - \vec{r}_i \right] \tag{2}$$

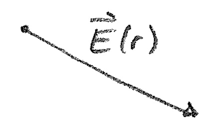
$\vec{E}_i(\vec{r})$ is the Coulomb electrostatic field at $\vec{r}$ due to charge $Q_i$ at $\vec{r}_i$



<u>Note:</u> As discussed in an earlier lecture, Eq. (1) <u>COMES FROM PHYSICS AND NOT FROM MATHEMATICS</u>! The linear superposition of fields/forces comes from physical experiments.




Also, as discussed in Lecture 19, let S be a smooth surface that encloses (but does not intersect with) the charges $Q_1 \ldots Q_N$.

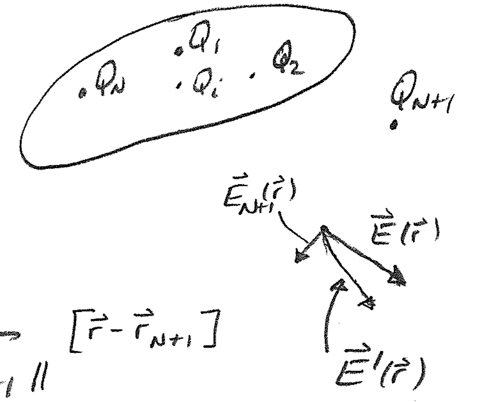



Then the total outward flux of $\vec{E}(\vec{r})$ through surface S is

$$\iint_S \vec{E} \cdot \hat{N}\, dS = \frac{1}{\varepsilon_0} Q \quad \text{where} \quad Q = \sum_{i=1}^{N} Q_i \qquad (3)$$

total charge enclosed by surface S

What if we added a charge $Q_{N+1}$, but located outside the surface S? It would certainly contribute to the total electrostatic field vector $\vec{E}(r)$: The new electrostatic field vector would be




$$\vec{E}'(\vec{r}) = \vec{E}(\vec{r}) + \vec{E}_{N+1}(\vec{r})$$

where $$\vec{E}_{N+1}(\vec{r}) = \frac{q_{N+1}}{4\pi\varepsilon_0} \frac{1}{\| \vec{r} - \vec{r}_{N+1} \|} [\vec{r} - \vec{r}_{N+1}] \qquad (4)$$

But what would be the total outward flux of the new field vector $\vec{E}'(\vec{r})$ through surface $S$? The answer is still $\frac{1}{\varepsilon_0} Q$ where $Q = \sum_{i=1}^{N} Q_i$ is the total charge enclosed by surface $S$.

The reason is – this sounds rather trivial – that charge $Q_{N+1}$ lies outside the surface $S$. But let's return to <u>why</u> its contribution to the total outward flux is zero.

Total outward flux of $\vec{E}'(\vec{r})$ through $S$:

$$\iint_S \vec{E}' \cdot \hat{N}\, dS = \iint_S [\vec{E} + \vec{E}_{N+1}] \cdot \hat{N}\, dS$$

$$= \iint_S \vec{E} \cdot \hat{N}\, dS + \iint_S \vec{E}_{N+1} \cdot \hat{N}\, dS$$

$$= \frac{1}{\varepsilon} Q + 0 \qquad (5)$$

The fact that the second integral is zero comes from the Divergence Theorem!

$$\iint_S E_{N+1} \cdot \hat{N}\, dS = \iint_D \vec{\nabla} \cdot \vec{E}_{N+1}\, dV$$

$$= \iint_D 0\, dV \qquad (6)$$

We can apply the Divergence Theorem because

$$\vec{\nabla} \cdot \vec{E}_{N+1}(\vec{r}) = 0 \quad \text{for} \quad \vec{r} \neq \vec{r}_{N+1} \qquad (7)$$

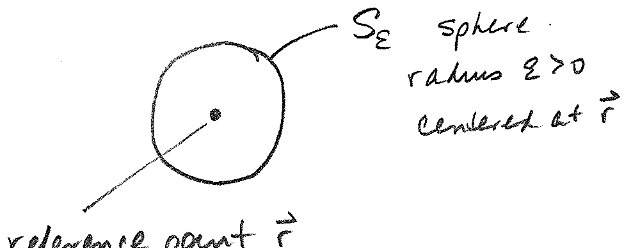
The singularity $\vec{r}_{N+1}$ does not lie inside surface S

This may seem like a moot point, but it will be important in the following discussion.

<u>Extension of the above result to continuous charge distributions</u>

We now assume that the electric charge is no longer concentrated at points (which implied infinite charge densities at these points). Instead, we assume that charge is "smeared out" over space according to a continuous charge distribution $\rho(\vec{r}) \geq 0$. As discussed before, this is the "continuum" description or model of matter at a macroscopic level. The actual (electrons) charges are so small in spatial extent as well as in actual magnitude of the charges, and there are usually so many of them per unit volume of material (eg. $cm^3$) that a "smeared out" picture is adequate.

We may view the charge density function $\rho(\vec{r})$ as follows:




Let $\Delta q$ be the amount of charge in $S_\varepsilon$

$\Delta V = \frac{4}{3}\pi\varepsilon^3$ volume of region enclosed by $S_\varepsilon$

then $$\rho(\vec{r}) = \lim_{\varepsilon \to 0} \frac{\Delta q}{\Delta V} = \lim_{\varepsilon \to 0} \frac{\Delta q}{\frac{4}{3}\pi\varepsilon^3}$$ provided that the limit exists, which we assume to be the case

This implies that

$$\rho(\vec{r}) = \frac{dq}{dV} \tag{8}$$

Alternatively, we say that the infinitesimal amount of charge $dq$ in an infinitesimal volume element $dV$ situated at $\vec{r}$ is

$$dq = \rho(\vec{r})\, dV \tag{9}$$

You may recall that the same idea was used for mass density functions.

Note that this implies that the <u>actual amount of charge at a point $\vec{r}$ is zero</u>! <u>There are no point charges!</u> (At least in this "continuum approximation".)

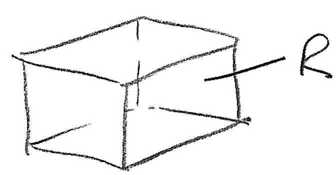
On that note, let's go back to the idea of a point charge $Q>0$ situated <u>at a point</u> $\vec{a}$. In this case, the density function $\rho(\vec{r}) = 0$ for $\vec{r} \neq \vec{a}$. Since there is no charge except at $\vec{a}$. At $\vec{a}$, however, using the previous definition

$$\rho(\vec{a}) = \lim_{\varepsilon \to 0} \frac{\Delta q}{\Delta V} = \lim_{\varepsilon \to 0} \frac{Q}{\frac{4}{3}\pi \varepsilon^3} = \infty \; !$$

Point charges correspond to infinite charge densities!

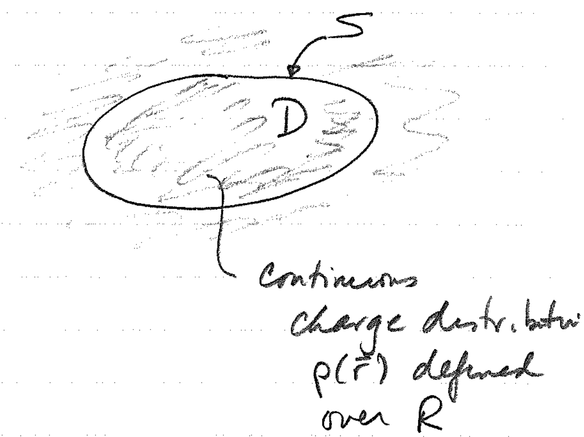
So why do we use them? Sometimes, it's because they are convenient. And they yield equations for forces, etc, that are easier to handle - eg. Coulomb law for electrostatic force. But there is a price to pay, especially if we wish to prove some mathematical results - recall the divergence theorem when singularities are involved

OK, back to continuous distributions of charges. We suppose that the charge is distributed over some region $R \subset \mathbb{R}^3$ - a large rectangular box, for example. (The details of $R$ are not important - it could represent a (finite) electrical wire, metal sheet, box, etc..)



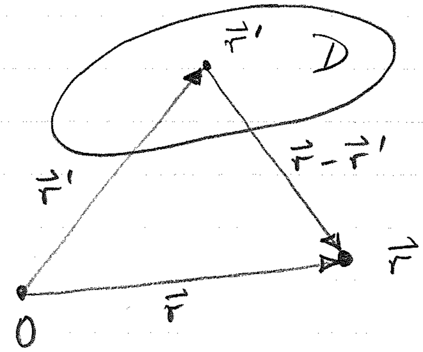

The important point is that $R$ is the domain of definition of $\rho(\vec{r})$ - $\rho(\vec{r})$ could be zero at places so that the charge is confined well within the boundary of $R$.

Now let $D \subset R$ be an arbitrary region with boundary $\partial D = S$ where $S$ is a piecewise smooth, orientable surface that satisfies the conditions of the Divergence Theorem.




We shall now refer to points in region $D$ with primed coordinates, e.g. $\vec{r}' = (x', y', z')$

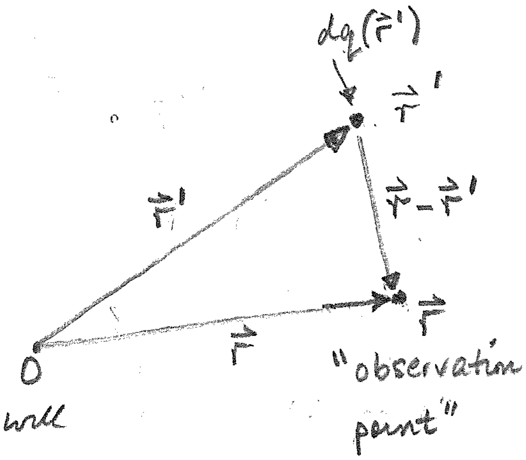
At each point $\vec{r}' \in D$, there is an infinitesimal volume element $dV$ that contains an infinitesimal amount of charge,




$$dq(\vec{r}') = \rho(\vec{r}')\, dV.$$

Now let $\vec{r} \in R$ be an "observation point" that may or may not be in region $D$ - in the diagram above, we have it sitting outside $D$

This entire conglomeration of charge, as defined by the density $\rho(\vec{r})$ will produce a net electrostatic field $\vec{E}(\vec{r})$ at each point $\vec{r} \in \mathbb{R}$. It may be viewed the net sum - actually integration - of all elements of charge $dq(\vec{r}')$ situated at all points $\vec{r}'$ of $R$ - we use primed coordinates to differentiate the positions of the charges $dq(\vec{r}')$ that will contribute to the field at an "observation point" $\vec{r}$




And what is the contribution of charge $dq(\vec{r}')$ to the total field $\vec{E}(\vec{r})$ at $\vec{r}$? From Coulomb's Law, it will be the infinitesimal field vector

$$\underset{\substack{\text{the components are} \\ \text{functions of } \vec{r}' \text{ and } \vec{r}}}{dE(\vec{r}';\vec{r})} = \frac{\overset{\text{charge}}{dq(\vec{r}')}}{4\pi\varepsilon_0} \frac{1}{\underbrace{\|\vec{r}-\vec{r}'\|}_{\text{separation}}{}^3} \left[\vec{r}-\vec{r}'\right] \qquad (10)$$

Here, the charge producing the field $d\vec{E}$ is $dq(\vec{r}')$, the distance from $\vec{r}$ to the charge at $\vec{r}'$ is $\|\vec{r}-\vec{r}'\|$, and the vector from $\vec{r}'$ to $\vec{r}$ is $\vec{r}-\vec{r}'$. In the past, We have considered charges $Q$ at the origin $\vec{O}$: Now, we must shift the location of the charge to $\vec{r}\,'$

So what is the net field $\vec{E}(\vec{r})$ at $\vec{r}$? We have to integrate over all elements of charge $dq(\vec{r}\,')$, $\vec{r}\,' \in R$. We can write this symbolically as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_R \frac{dq(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|^3}\,[\vec{r}-\vec{r}\,']$$

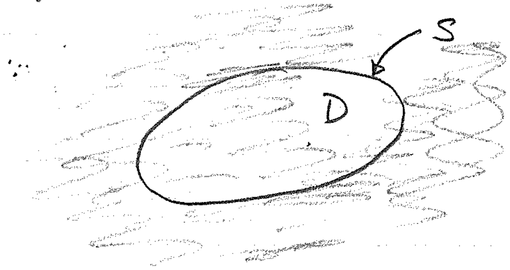
$$= \frac{1}{4\pi\varepsilon_0} \iiint_R \frac{\rho(\vec{r}\,')\,d\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3}\,[\vec{r}-\vec{r}\,'] \qquad (11)$$

This is actual a vector of integrals - each component of the vector will involve $\underbrace{x', y', z'}_{\vec{r}\,'}$ as well as $\underbrace{x, y, z}_{\vec{r}}$

There may be one question: What happens when $\vec{r}\,' = \vec{r}$, i.e we integrate over the observation point? At that point, we consider the charge element $dq(\vec{r})$ at $\vec{r}$, so the distance from the observation point $\vec{r}$ to the charge $dq(\vec{r})$ is ZERO! One might suspect that there will be a "blow up". Indeed, there would be, if the charge $dq(\vec{r})$ were a point charge, in which case the density $\rho(\vec{r})$ would be infinite. But the saving grace is that the density is finite. The volume element $dV' = d\vec{r}'$ around $\vec{r}$ actually "cancels" the $\frac{1}{\|r - \vec{r}'\|^3}$ term in the denominator during the integration. We'll simply bypass any further discussion of the technicalities behind this.

OK, that's all fine, but is there an easier way to determine the field vector $\vec{E}(\vec{r})$ than the integration in Eq. (11). Thanks to Vector Calculus, in particular, surface integrals and the divergence Theorem, there is, and we outline it below.

Once again, let $D \subset R$ be an arbitrary region with boundary $\partial D = S$ where $S$ is a piecewise smooth, orientable surface that satisfies the Divergence Theorem.




Let's now return to the (unknown) electrostatic field $\vec{E}(\vec{r})$ which is produced by <u>all</u> charge in region $R$. Using the same arguments as employed in the case of discrete charges, we can state that the <u>total</u> <u>outward</u> <u>flux</u> of $\vec{E}$ through surface $S$ is

$$\iint_S \vec{E} \cdot \hat{N} \, dS = \frac{Q(D)}{\varepsilon_0} \qquad (12)$$

where $Q$ is the total charge (in region $D$) enclosed by surface $S$. The reason, once again: Any charge element $d\vec{q}(\vec{r}\,')$ lying outside surface $S$ will <u>not</u> contribute to the outward flux since the divergence of the field that this element produces is <u>zero</u>. Eq. (12) is the continuous version of "Gauss' Law" for electrostatics.

But if we know the charge density $\rho(\vec{r})$, we know the total charge $Q(D)$: It is, by definition,

$$Q(D) = \iiint_D \rho(\vec{r})\, dV \qquad (13)$$

Let's combine (12) & (13)

$$\iint_S \vec{E} \cdot \hat{N}\, dS = \frac{1}{\varepsilon_0} \iiint_D \rho(\vec{r})\, dV \qquad (14)$$

But this is "apples = oranges" The LHS is a surface integral a & the RHS is a volume integral. We now use Gauss' Divergence Theorem to express the LHS surface integral as a volume integral, i.e

$$\iint_S \vec{E} \cdot \hat{N}\, dS = \iiint_D \vec{\nabla} \cdot \vec{E}\, dV \qquad (15)$$

But can we really do this? Remember that we could not do this for point charges, where the charge density functions $\rho(\vec{r})$ were infinite at points.

We'll answer this by simply making the assumption that the field vector $\vec{E}(\vec{r})$ is not only continuous but also $C^1$, so that the divergence $\vec{\nabla}\cdot\vec{E}$ is continuous. Is this justified, even though Coulomb's Law for infinitesimal charges holds? The answer is "Yes", but we avoid the details - we'll simply state that the results obtained at the end are consistent with electrostatics.

Let's now proceed by equating the RHS of (14) and (15):

$$\iiint_D \vec{\nabla}\cdot\vec{E}\, dV = \frac{1}{\varepsilon_0}\iiint_D \rho(\vec{r})\, dV \qquad (16)$$

We rewrite this equation as follows

$$\iiint_D \left[\vec{\nabla}\cdot\vec{E}(\vec{r}) - \frac{1}{\varepsilon_0}\rho(\vec{r})\right] dV = 0 \qquad (17)$$

We now "invoke the same wand" as we did in the previous lecture on conservation laws:

1. Continuity of integrand at all $\vec{r}\in R$
2. Arbitrariness of region $D$

Conclusion: The electrostatic field $\vec{E}(\vec{r})$ produced by the charge distribution $\rho(\vec{r})$, $\vec{r} \in R$, satisfies the equation

$$\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\varepsilon_0} \rho(\vec{r}) \qquad (18)}$$

This is known as the "differential form of Gauss' Law" or "<u>Maxwell's First Equation for Electrostatics.</u>"

Let's notice a couple of consequences of this equation

1. If $\rho(\vec{r}) = 0$, then $\vec{\nabla} \cdot \vec{E}(\vec{r}) = 0$

   In other words, no charge at $\vec{r}$ $\Rightarrow$ no divergence of $\vec{E}$ at $\vec{r}$

2. If $\rho(\vec{r}) > 0$, then $\vec{\nabla} \cdot \vec{E}(\vec{r}) > 0$

<u>"Where there is charge, there is divergence"</u>

(You may remember me saying this a long time ago.)

There still remains the question of how to solve for $\vec{E}(\vec{r})$. Technically, Eq. (18) is a partial differential equation in the unknown $\vec{E}(\vec{r})$. But it is one equation involving three derivatives, i.e $\frac{\partial E_1}{\partial x}, \frac{\partial E_2}{\partial y}, \frac{\partial E_3}{\partial z}$.

A considerable simplification is afforded by the fact that $\vec{E}(\vec{r})$ is conservative, i.e

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) \qquad (19)$$

where $V(\vec{r})$ is the electrostatic potential associated with $\vec{E}(\vec{r})$. Substituting this result into (18):

$$\vec{\nabla} \cdot \left(-\vec{\nabla} V(\vec{r})\right) = \frac{1}{\varepsilon_0} \rho(\vec{r})$$

$$\Rightarrow \quad -\vec{\nabla} \cdot \vec{\nabla} V(\vec{r}) = \frac{1}{\varepsilon_0} \rho(\vec{r})$$

or

$$\vec{\nabla} \cdot \vec{\nabla} V(\vec{r}) = -\frac{1}{\varepsilon_0} \rho(\vec{r})$$

But the LHS is the Laplacian of $\vec{E}$

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{1}{\varepsilon_0} \rho(\vec{r})} \qquad (20)$$

Eq. (17) is known as "Poisson's Equation". In the absence of charge, ie $\rho(\vec{r}) = 0$ for $\vec{r} \in D$, Eq. (18) becomes

$$\nabla^2 V(\vec{r}) = 0 \qquad \vec{r} \in D \qquad (19)$$

which is known as Laplace's equation.

Both Eq. (18) and (19) are linear second order partial differential equations in the scalar-valued function $V(\vec{r}) = V(x,y,z)$ $\big($or $V(r,\theta,\varphi)\big)$

Because of their importance in many areas of science and engineering, a great deal of effort was spent by mathematicians over the past three centuries, especially in the 1700's - 1900's. You will encounter these equations in a number of future courses in electricity and magnetism, differential equations (of course!), continuum mechanics, including fluid mechanics.