Lecture 19

Chaotic dynamics (cont'd)

Spatial distribution of chaotic orbits, "visitation frequencies", invariant measures

Suppose that we are told that a function $f : I \to I$ demonstrates chaotic behaviour on I, that is, for "almost all" seed points $x_0 \in I$, the orbit of x_0, given by the iteration procedure,

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{1}$$

exhibits seemingly random behaviour when the x_n are plotted vs. n. Moreover, the orbit of x_0 is dense on I: Given any point $a \in I$, and any neighbourhood N_δ of a, i.e., the interval $(a - \delta, a + \delta)$, there is an element x_n to be found in N_δ.

If this were all the information that could be obtained from the iterates $\{x_n\}$ defined in (1), then there wouldn't seem to be a way to tell whether a chaotic sequence was generated from a function $f : I \to I$ or another function, say, $g : I \to I$. In other words, we wouldn't be able to tell the difference between a chaotic orbit generated by the iteration of the Tent Map and a chaotic orbit generated by iteration of the logistic map $f_4(x)$.

In this section, we show very briefly that there are differences between the two sets of chaotic orbits, even though they are both **dense** on I. **To see these differences, we look at how the iterates $\{x_n\}$ are distributed over the interval I.**

For example: Do the iterates $\{x_n\}$ tend to be spread out rather evenly over the interval I? Or are they somewhat "concentrated" at some parts of I and less concentrated at other parts. This question has been at the heart of an enormous amount of research in dynamical systems theory over the past half-century and more. Here we provide a small glimpse into the subject.

There is a relatively simple (apart from some possible problems due to finite precision) numerical procedure to visualize how the iterates x_n defined in (1) are distributed over the interval I. In what follows, we let $I = [a, b]$. (Typically, $I = [0, 1]$, but we'll keep the discussion general.)

Step No. 1: For an N relatively large (say 1000-10000), form a partition of the interval $[a, b]$ in the manner done in first-year Calculus, i.e., let

$$\Delta t = \frac{b - a}{N}, \tag{2}$$

and define

$$t_k = a + k\,\Delta t\,, \qquad 0 \le k \le N\,, \tag{3}$$

so that $t_0 = a$ and $t_N = b$.

Step No. 2: This partition will define a set of N subintervals of I,

$$I_k = [t_{k-1}, t_k]\,, \quad 1 \le k \le N\,. \tag{4}$$

For reasons that will become clear below, define the following set of half-open intervals,

$$J_k = [t_{k-1}, t_k)\,, \quad 1 \le k \le N-1\,, \tag{5}$$

along with the final interval,

$$J_N = [t_{N-1}, t_N]\,. \tag{6}$$

In the special case that $I[0,1]$, for which $a = 0$ and $b = 1$,

$$\Delta t = \frac{1}{N}\,, \tag{7}$$

and

$$t_k = k\Delta t = \frac{k}{N}\,, \tag{8}$$

with $p_0 = 0$ and $p_1 = 1$.

Step No. 3: Initialize a "counting vector," – call it $\mathbf{c}$, with N elements, so that

$$c_k = 0\,, \quad 1 \le k \le N\,. \tag{9}$$

The entries of $\mathbf{c}$ will be integers.

Step No. 4: Now choose a "good" seed point $x_0 \in I$, i.e., a point that is not preperiodic (or at least hopefully not preperiodic). Start computing the elements $\{x_n\}$ of the forward orbit of x_0 using (1), i.e.,

$$x_{n+1} = f(x_n)\,, \quad n \ge 0\,. \tag{10}$$

This will involve some kind of "loop" in your computer program. After you have computed each iterate x_n, determine the particular interval J_k, in which x_n lies. This can done in the following way (or some slight modification of it):

$$k = \mathrm{int}\left[\frac{1}{\Delta t}(x_n - a)\right] + 1\,, \tag{11}$$

where, for a $y \in \mathbb{R}$,

$$\text{int}[y] = \text{integer part of } y\,. \tag{12}$$

Rationale: If x_n lies in J_k, then

$$\begin{aligned} t_{k-1} \le x_n < t_k \quad &\Longrightarrow \quad a + (k-1)\Delta t \le x_n < a + k\Delta t \\ &\Longrightarrow \quad (k-1)\Delta t \le x_n - a < k\Delta t \\ &\Longrightarrow \quad (k-1) \le \frac{1}{\Delta t}(x_n - a) < k \end{aligned} \tag{13}$$

This implies that

$$k - 1 = \text{int}\left[\frac{1}{\Delta t}(x_n - a)\right]\,, \tag{14}$$

which yields (12).

After determining "k", the index of the interval I_k in which x_n lies, increase the appropriate entry of $\mathbf{c}$ by one, i.e.,

$$c_k = c_k + 1\,. \tag{15}$$

Step No. 5: Perform the iteration procedure in (10) for a sufficiently large number M of times, say $M = 50,000$, or, better yet, $M = 100,000$, or even $M = 10^6$. The larger the better: These computations do not take a lot of time.

At the end of the computation, you will have produced an N-vector, $\mathbf{c}$. Hopefully, some, if not all, of its entries c_k will be nonzero.

Question: What is this vector $\mathbf{c}$?

Answer: Each element c_k of this vector for $1 \le k \le N$, has recorded **the number of times that the interval** $I_k = [t_{k-1}, t_k)$ **has been visited** by an iterate x_n over the orbit $1 \le n \le M$.

If you now plot the values of the elements c_k vs. k, you will get an idea of how the iterates x_n are distributed over the interval I. Intervals I_k with higher numbers of visitation will have higher counts c_k.

In order to be able to compare the results of this counting experiment for different choices of M, the total number of iterates computed, it is convenient to **normalize** the count vector $\mathbf{c}$ by defining the following N-vector $\mathbf{p}$, the elements of which will not be integers, but fractions:

$$p_k = \frac{c_k}{M} \quad 1 \le k \le N\,. \tag{16}$$

Technically, we should write $p_k(M)$, since our values of p_k will depend on M, but we leave the M out for the moment. Note that $0 \le p_k \le 1$ for $1 \le k \le N$, and

$$\sum_{k=1}^{N} p_k = 1\,. \tag{17}$$

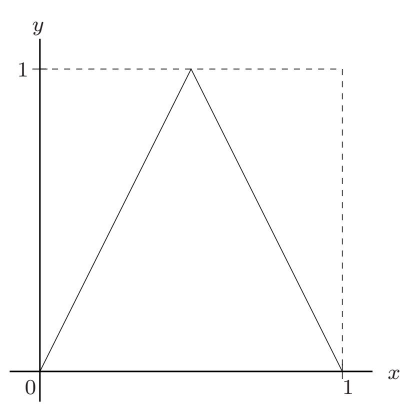
Each element p_k, $1 \le k \le N$, may be interpreted in at least two ways, which are not unrelated:

1. p_k is the **fraction of the iterates** $\{x_n\}_{n=1}^{M}$ that have visited interval J_k. If we view the set of iterates $\{x_n\}_{n=1}^{M}$ as a huge collection of numbers between 0 and 1, the p_k indicate how they are distributed in the N bins J_k, $1 \le k \le N$. This is, of course, a discrete approximation to how they are distributed over $[0, 1]$.

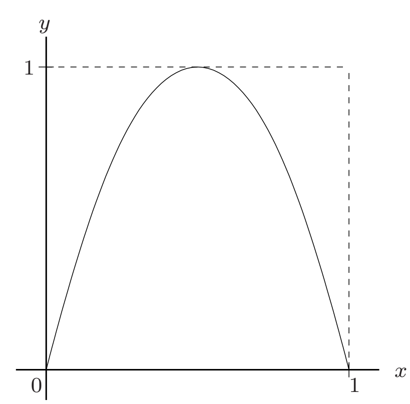
2. p_k is the **visitation frequency** of interval J_k – or at least an approximation of it – by the iterates $\{x_n\}_{n=1}^{M}$. This has a probabilistic interpretation: For each $n > 0$, we may view p_k as the **probability** that iterate x_n will be found in interval J_k.

The term "visitation frequency" sounds "statistical" and suggests that the results of our computations are approximations to a true "visitation frequency" that is obtained by letting the number of iterations $M \to \infty$.

Before going on, let us examine the results of a few computations for two chaotic maps on $[0, 1]$ that we have studied to date: (i) the Tent Map and (ii) the logistic map $f_4(x)$, the graphs of which are once again plotted below:



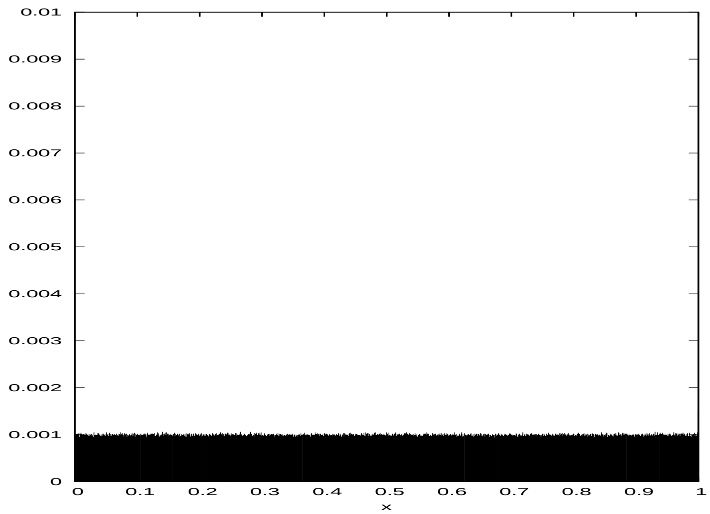
Tent Map $T(x)$.



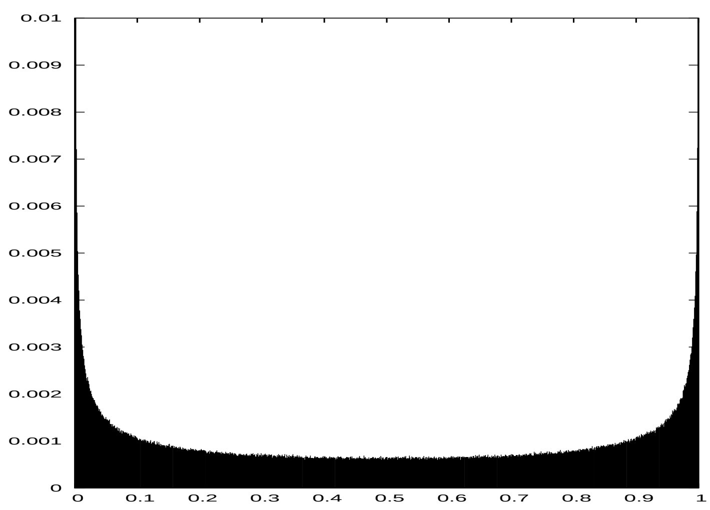
Logistic map $f_4(x) = 4x(1 - x)$.

In the plots on the next page are shown plots of the elements, p_k, $1 \leq k \leq M$, of the vector $\mathbf{p}$ obtained when the interval $[0, 1]$ is divided into $N = 1000$ subintervals and $M = 2 \times 10^6$ iterates are used. The most noticeable difference between the two plots is that one (the tent map) is quite "flat" compared to the other (logistic-4).

Approximations to visitation frequencies for two chaotic maps on [0,1]



Tent Map $T(x)$



Logistic map $f_4(x) = 4x(1 - x)$

In both cases, $M = 1000$ bins and $N = 2 \times 10^6$ iterates were used.

With regard to the tent map case, note that the value of each p_k is roughly 0.001, i.e.,

$$p_k \approx 0.001 = \frac{1}{N}. \tag{18}$$

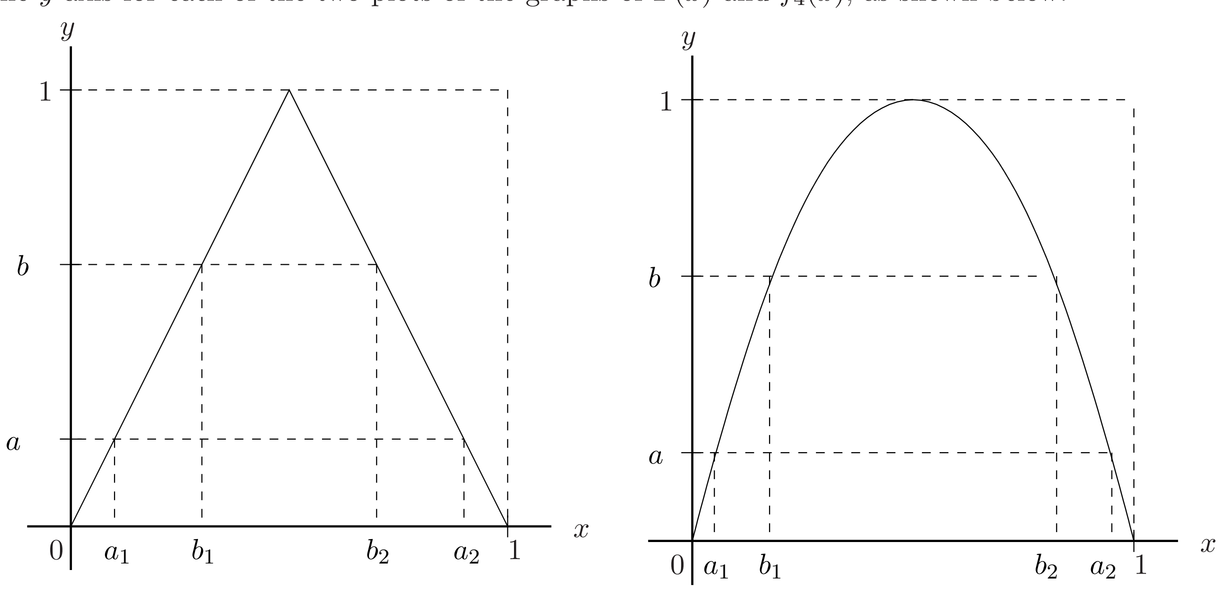
This is consistent with Eq. (17) and indicates that the iterates are visiting the entire interval [0,1] in a quite uniform manner.

We might not be so surprised to see that the distribution associated with the tent tap is "flat". After all, the tent map is piecewise linear, i.e., the "pieces" are straight lines, as oppposed to the logistic function $f_4(x)$, which is "curved." This indeed has something to do with the flatness of the tent map case, and we'll "prove" that the distribution is flat, i.e., uniform.

The question then remains, "Why does the distribution for the logistic-4 map curl upwards near the ends of the interval, implying that these outer regions are visited more frequently than the inner region around $x = \frac{1}{2}$?" An explanation is now to be provided.

In order to understand the shapes of these visitation frequency plots, we shall have to make use of a kind of "conservation law", that acknowledges the 2-to-1 nature of the tent and logistic-4 maps. (The idea extends in a straightforward manner to other "many-to-one" maps, e.g., 3-to-1 maps.)

Consider an interval $[a, b] \subset [0, 1]$. Instead of being concerned where points from $[a, b]$ are **going** under the action of a map f (tent or logistic-4), we are going to be concerned about what points from [0,1] are **coming** to $[a, b]$ under the action of f. It is therefore convenient to place the interval $[0, 1]$ on the y-axis for each of the two plots of the graphs of $T(x)$ and $f_4(x)$, as shown below.



Tent Map $T(x)$.

Logistic map $f_4(x) = 4x(1 - x)$.

In each figure are shown the two "preimages" of the interval $[a, b]$ under the action of the map concerned:

- Tent map: $T([a_1, b_1]) = [a, b]$ and $T([b_2, a_2]) = [a, b]$.

- Logistic-4 map: $f_4([a_1, b_1]) = [a, b]$ and $f_4([b_2, a_2]) = [a, b]$.

Note that for both maps, the interval $[b_2, a_2]$ is "flipped," due to the decreasing nature of the map for $\frac{1}{2} < x \leq 1$.

We now develop our "conservation law," which is essentially a law of probabilities. We'll be working in the discrete framework introduced earlier, that is, dividing the interval $[0, 1]$ into N subintervals J_k. Associated with each interval J_k is a probability p_k that the iterate will visit it during an orbit of length M. We'll also assume that N is sufficiently (enormously?) large so that the discrete approximations that we are employing are sufficiently accurate. We'll also assume that the length M of the orbit is large/enormous, essentially approaching the limit $M \to \infty$. Later, we shall actually let N, the number of subintervals, go to infinity in order to arrive at a continuous description of the visitation frequency which will then employ the differential dx instead of the bin width $\Delta x = \Delta t$.

In what follows, we shall, for the most part, adopt the first of the two interpretations of the quantities p_k, $1 \leq k \leq N$, defined in Eq. (16), that is, that each p_k is the **fraction of iterates** $\{x_n\}_{n=1}^{M}$ found in interval J_k. We use this interpretation to adopt the next set of assumptions:

1. Let $K \subseteq [0, 1]$ be an interval, and suppose that K is the union of a number of subintervals J_k, i.e.,

$$K = \bigcup_{k=k_1}^{k_2} J_k \,. \tag{19}$$

   This is equivalent to the statement,

$$K = [t_{n_1-1}, t_{n_2}] \,, \tag{20}$$

   where the t_k are the partition points defined earlier. Then the **fraction of iterates** $\{x_n\}_{n=1}^{M}$ that have visited interval K, to be denoted as $F(K)$ is given by

$$F(K) = \sum_{k=n_1}^{n_2} p_k \,. \tag{21}$$

   Special case: When $K = [0, 1]$, then $n_1 = 1$ and $n_2 = N$ so that the sum in (21) is 1, as it should be: The fraction of all iterates that lie in [0,1] is 1, since all iterates $x_n \in [0, 1]$.

   Note: Eq. (21) is often written in the more convenient form,

$$F(K) = \sum_k{}' p_k \,, \tag{22}$$

where the prime on the summation indicates that the summation is over only those $k \in \{1, 2, \cdots N\}$ for which $J_k \subseteq K$. Or, to remove any confusion, we could write,

$$F(K) = \sum_{\{k | J_k \subseteq K\}} p_k \,. \tag{23}$$

2. **This one is very important!** With reference to the graphs of the Tent Map and the Logistic-4 Map on the previous page, the fraction of iterates that lie in the interval $[a, b]$ is equal to the sum of the fractions of iterates lying in $[a_1, b_1]$ and $[b_2, a_2]$. Mathematically,

$$F([a, b]) = F([a_1, b_1]) + F([b_2, a_2]) \,. \tag{24}$$

This is a kind of **conservation of iterates** (which is essentially a conservation of mass).

The iterates in $[a, b]$ come from the two intervals $[a_1, b_1]$ and $[b_2, a_2]$. If we have arrived at a kind of stationary distribution that tells us how the iterates are distributed over the intervals, then the above conservation law has to hold. Later, we shall state this law mathematically.

Note: We have relied on the assumption that Δt, the length of the subintervals J_k, is sufficiently small so that the intervals involved above, i.e., $[a, b]$, $[a_1, b_1]$ and $[b_2, a_1]$, can be expressed – or at least well approximated – as unions of the subintervals J_k, i.e., no subintervals J_k have been "split". In the limit $N \to \infty$, these approximations will become exact and Eq. (24) is valid without the use of the subintervals J_k.

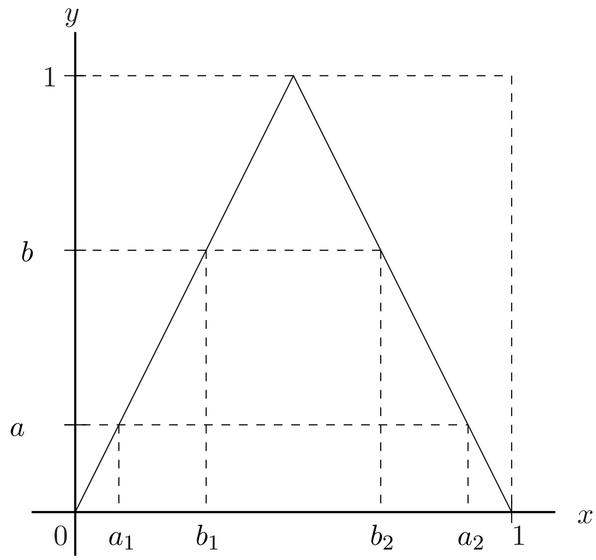
We are now in a position to argue – not prove, but at least understand – why the distribution for the Tent Map is "uniform," i.e., all of the probabilites p_k are constant and equal $\frac{1}{N}$. It is, indeed, because of the fact that the two components of $T(x)$ are straight as well as having slopes of equal magnitude, namely, 2. By simple geometry, the lengths of the intervals $[a_1, b_1]$ and $[b_2, a_2]$ are equal and one-half the length of the interval $[a, b]$ as shown once again below.

It should be fairly easy to see that if the conservation equation in (24) holds for **any** interval $[a, b] \subseteq [0, 1]$, then the p_k should all be equal, i.e.,

$$p_k = \frac{1}{N} \,, \quad 1 \leq k \leq N \,. \tag{25}$$

This is often referred to as a **uniform distribution**.

And what about the distribution associated with the Logistic-4 map? Why does it increase as we approach 0 and 1? Very loosely speaking, because the magnitude of the tangents to the graph of $f_4(x)$, i.e., $|f'(x)|$, increases as $x \to 0^+$ and $x \to 1^-$ and decreases as $x \to \frac{1}{2}$



Tent Map $T(x)$.

Looking at the graph of $f_4(x)$ with the interval $[a, b]$ on the y-axis and its preimages, $[a_1, b_1]$ and $[b_2, a_1]$, note that the lengths of these intervals are shorter than one-half the length of $[a, b]$. This is due to the fact that the magnitudes of the tangents to the graph of $f_4(x)$, i.e., $|f'(x)|$ increase as $x \to 0^+$ and $x \to 1^{-1}$, approaching the value of 4 in the limits. As $[a, b]$ is moved downward toward the origin, the lengths of these two preimages gets even smaller. Loosely speaking (or writing), in order for the conservation equation in (24) to hold, the fractions of the iterates on these two intervals has to be greater than the uniform distribution in (25).

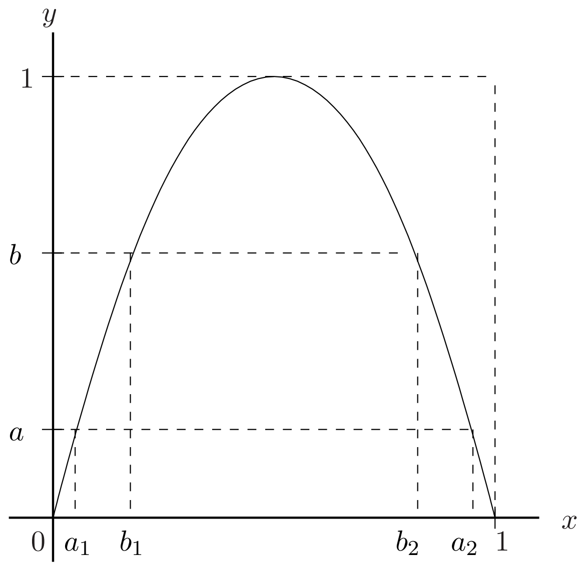
Admittedly, these are very "loose" or "heuristic" descriptions of why we expect the shape of the distributions of the p_k fractions to be what they are. Let us now go to a more mathematical description.

In order to do so, let us first consider the limit $M \to \infty$, where M is the number of iterates. As M increases, the number of iterates falling in the interval J_k, which we have called c_k, will generally increase with M. In fact, we should denote this number of iterates as $c_k(M)$. And we should once again acknowledge that the fractions p_k are also functions of M and write

$$p_k(M) = \frac{c_k(M)}{M} \,. \tag{26}$$

We now claim that the following limits exist,

$$\lim_{M\to\infty} p_k(M) = \lim_{M\to\infty} \frac{c_k(M)}{M} = p_k \,, \quad 1 \le k \le N \,. \tag{27}$$



Logistic Map $f_4(x)$.

Then p_k is the **limiting fraction of iterates** that are found in subinterval J_k as we let the number M of iterates go to infinity. Once again,

$$\sum_{k=1}^{N} p_k = 1\,. \tag{28}$$

We now consider the p_k to define a **piecewise constant** function $P(x)$ on $[0,1]$:

$$P_N(x) = p_k \quad \text{if} \quad x \in J_k\,, \quad 1 \leq k \leq N\,. \tag{29}$$

The subscript N reminds us that N subintervals J_k are used in the construction of this function.

Now we do something that will seem quite strange: We define a new function $\rho_N(x)$ as follows,

$$\rho_N(x) = \frac{1}{\Delta t} P_N(x) = \frac{p_k}{\Delta t} \qquad \text{if} \quad x \in J_k\,, 1 \leq k \leq N\,. \tag{30}$$

This must seem very strange, indeed, since Δt is very small, and since we eventually wish to take the limit $N \to \infty$, which implies that $\Delta t \to 0$. But as N, the number of intervals, increases, and Δt decreases, each p_k decreases – there are more intervals in which to find the iterates! The reason we define $\rho_N(x)$ in Eq. (30) is that the fraction $F(J_k)$ of iterates found in subinterval J_k is now given by

$$F(J_k) = p_k = \rho_N(x_k^*)\Delta\, x\,, \quad x_k^* \in J_k\,, \tag{31}$$

where $\Delta x = \Delta t$ (just to keep everything in terms of x). Note that x_k^* can be any point in J_k, since $F(x)$ is constant over each subinterval J_k.

The fraction of iterates found in an interval $K = [a, b] \subset I$ now becomes

$$F(K) = \sum_k{}' \rho_N(x_k^*)\, \Delta x\,, \quad x_k^* \in J_k\,, \tag{32}$$

where, once again, the summation is performed only over those indices $k \in \{1, 2, \cdots, N\}$ such that $J_k \subseteq K$.

Note: Is this starting to look like something from first-year Calculus, i.e., Riemann integation?

> **Another note:** We may view $\rho_N(x)$ in Eq. (30) as a **density function**, i.e., **the (normalized) number of iterates per unit length**. We write "normalized" since the total number of "normalized" iterates over the entire interval $[a, b]$ is 1, i.e.,
>
> $$\sum_{k=1}^{N} p_k = 1\,. \tag{33}$$
>
> In this way, one could think of the iterates as representing electric charges, in which case $\rho_N(x)$, $x \in J_k$, is the **lineal charge density** (charge per unit length) over the interval J_k.

We now perform the limiting operation $N \to \infty$. In this limit, Δx, the length of the subintervals J_k, will go to zero. The summation over these subintervals of length Δx will become an integration over the differential dx. We claim that in the limit $N \to \infty$, the piecewise constant functions $\rho_N(x)$ converge to a function $\rho(x)$, for $x \in [0, 1]$. For any subinterval $[a, b] \subseteq [0, 1]$, the limiting fraction of iterates in $[a, b]$ is **no longer a summation over all subintervals** J_k lying in $[a, b]$ as done in Eq. (21) **but rather an integration** over the interval $[a, b]$, i.e.,

$$F([a, b]) = \int_a^b \rho(x)\, dx\,. \tag{34}$$

We have arrived at a continuous description of the fractional distribution of iterates over the interval $[0, 1]$. Note that the function $\rho(x)$ is a normalized distribution since

$$F([0, 1]) = \int_0^1 \rho(x)\, dx = 1\,. \tag{35}$$

Eq. (34) leads to the following continuous version of the conservation equation in (24): For any $[a, b] \subseteq K$,

$$\int_a^b \rho(x)\, dx = \int_{a_1}^{b_1} \rho(x)\, dx + \int_{b_2}^{a_2} \rho(x)\, dx\,. \tag{36}$$

We are now going to state this conservation result more generally as well as mathematically. In what follows, we let I denote an interval on which a function $f : I \to I$ is defined. f may or may not be chaotic. For any subset $S \subset I$, we define the following set,

$$f^{-1}(S) = \{x \in I \,, f(x) \in S\} \,. \tag{37}$$

In other words, $f^{-1}(S)$ is the set of all points in I that are mapped by f to the set S. In the case of each of the Tent and Logistic-4 maps, S is the interval $[a, b] \subset [0, 1]$ and

$$f^{-1}([a, b]) = [a_1, b_1] \cup [b_2, a_2] \,, \tag{38}$$

where the a_i and b_i depend on the maps. The above relation is true because

$$f([a_1, b_1]) = f([b_2, a_2]) = [a, b] \,. \tag{39}$$

Definition: Let I be an interval and $f : I \to I$. If there exists a function $\rho : I \to \mathbb{R}$ such that for all $S \subseteq I$,

$$\int_S \rho(x)\, dx = \int_{f^{-1}(S)} \rho(x)\, dx \,, \tag{40}$$

then ρ is said to be the **invariant (probability or normalized) density function** which defines the **invariant measure** associated with the mapping $f : I \to I$.

Notes:

- The notation $\int_{f^{-1}(S)}$ implies an integration over the entire set $f^{-1}(S)$ defined earlier. Eq. (36) stated earlier,

$$\int_a^b \rho(x)\, dx = \int_{a_1}^{b_1} \rho(x)\, dx + \int_{b_2}^{a_2} \rho(x)\, dx \,, \tag{41}$$

  is a special case of Eq. (40).

- The reason for the term **invariant measure** is that the density function ρ is considered to define a "measure" of subsets $S \subset I$ – a generalized notion of "length", a kind of "weighted length". Regions of I that have higher ρ-values, i.e., fractions of iterates, are weighted more heavily that those regions with lower ρ-values. Usually, the invariant measure associated with a dynamical system $f : I \to I$ is denoted as "μ". The invariant measure, or "μ-measure" of an interval $[a, b]$ is given by

$$\mu([a, b]) = \int_a^b \phi(x)\, dx \,, \tag{42}$$

which, as we saw earlier, is the fraction of iterates in the interval $[a, b]$. The conservation relation in (40) may be expressed as follows,

$$\mu(S) = \mu(f^{-1}(S)) \quad \text{for all} \quad S \in I\,. \tag{43}$$

- Our discussion of the density function $\rho(x)$ may also bring back memories of earlier courses in Calculus where the ideas of one-dimensional **mass** and **charge densities** were discussed, i.e., infinitesimal amount of mass or charge per unit length. For example, given a thin rod that lies along x-axis on the interval $[a, b]$, if $\rho(x)$ represents the mass density of a rod at point x, then the total mass of the rod is

$$M = \int_a^b \rho(x)\,dx\,. \tag{44}$$

The invariant density functions for the Tent and Logistic-4 maps

Tent Map $T(x)$

Here we simply state that, as expected, the invariant density function $\rho(x)$ for the Tent Map on $[0, 1]$ is a constant function – no regions have a higher fraction of iterates than others. In the case that $I = [0, 1]$,

$$\rho(x) = 1\,, \quad x \in [0, 1]\,, \tag{45}$$

Referring to the earlier figure which shows that graph of the Tent Map function $T(x)$ along with the interval $[a, b]$ and its two preimages, the conservation equation in (36) becomes

$$\int_a^b dx = \int_{a_1}^{b_1} dx + \int_{b_2}^{a_2} dx\,. \tag{46}$$

Let us finally state explicitly what the a_i and b_i are:

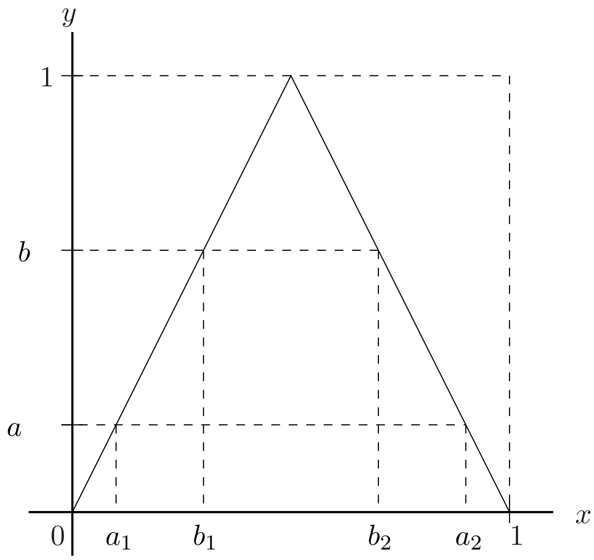
$$a_1 = \frac{1}{2}a\,, \quad b_1 = \frac{1}{2}b\,, \tag{47}$$

and

$$b_2 = 1 - \frac{1}{2}b\,, \quad a_2 = 1 - \frac{1}{2}a\,. \tag{48}$$

The integrals in (46) are, of course, simple to evaluate:

$$\begin{aligned} \text{LHS} &= b - a\,. \\ \text{RHS} &= (b_1 - a_1) + (a_2 - b_2) \\ &= \frac{1}{2}(b-a) + \frac{1}{2}(b-a) \\ &= b - a\,. \end{aligned} \tag{49}$$



Tent Map $T(x)$.

which is satisfied for all $[a, b] \in [0, 1]$.

The invariant measure μ defined by the density function $\rho(x) = 1$ is

$$\begin{aligned} \mu([a,b]) &= \int_a^b \rho(x)\,dx \\ &= \int_a^b dx \\ &= b - a\,. \end{aligned} \tag{50}$$

In this case, the μ-measure of the interval $[a, b]$ is the length of the interval, the usual notion of the "size" of an interval. This is somewhat of a coincidence since the interval I on which the Tent Map $T(x)$ is defined is $[0, 1]$. If we were to consider a tent map on $I = [0, 2]$, then the constant density function ρ would be

$$\rho(x) = \frac{1}{2}\,, \tag{51}$$

so that

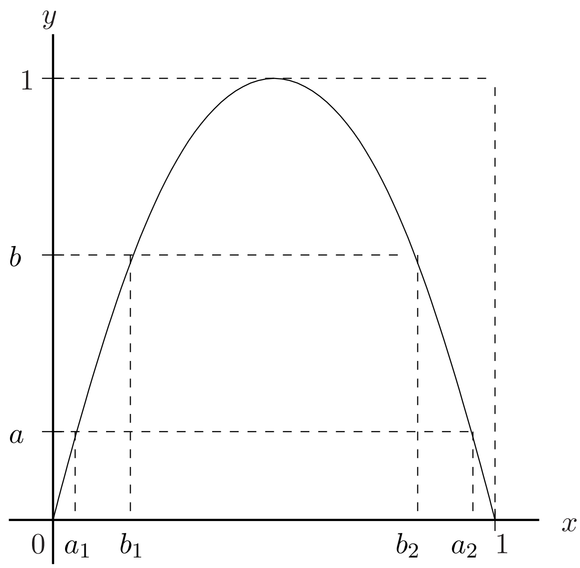
$$\int_I \rho(x)\,dx = \int_0^2 \frac{1}{2}\,dx = 1\,. \tag{52}$$

When the density function $\rho(x)$ is constant, the measure defined by ρ is often called **uniform measure**.

Logistic-4 map $f_4(x) = 4x(1-x)$

Referring to the earlier figure which shows the graph of the logistic-4 function $f_4(x)$ along with the interval $[a, b]$ and its two preimages, let us rewrite the conservation equation in (36) that would have to be solved by the invariant density function $\rho(x)$ associated with the $f_4(x)$ map.

$$\int_a^b \rho(x)\,dx = \int_{a_1}^{b_1} \rho(x)\,dx + \int_{b_2}^{a_2} \rho(x)\,dx\,. \tag{53}$$



Logistic Map $f_4(x)$.

The a_i and b_i are easily found to be as follows:

$$a_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1-a}\,, \quad b_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1-b}\,, \tag{54}$$

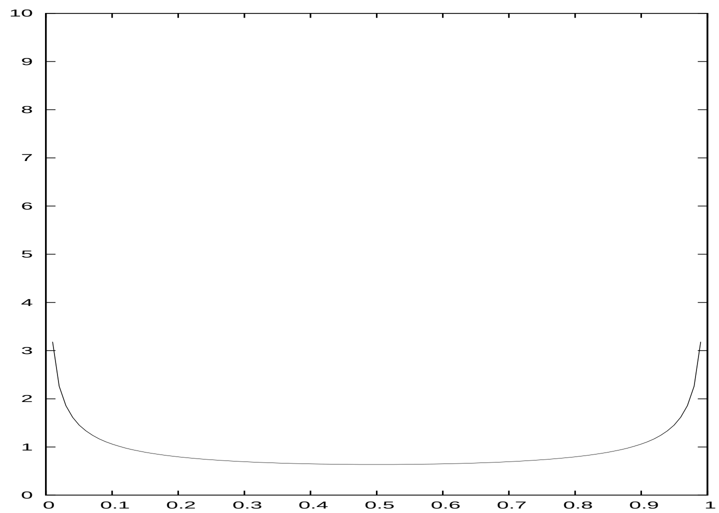
and

$$b_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-b}\,, \quad a_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-a}\,, \tag{55}$$

We now state a remarkable result – the density function $\rho(x)$ satisfying (53) with the a_i and b_i defined above, is known analytically:

$$\boxed{\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}}\,. \tag{56}$$

The graph of $\rho(x)$, presented in the figure below, demonstrates a strong similarity to the distribution of iterates of the logistic-4 map obtained numerically and presented earlier.



Plot of density function $\rho(x) = \dfrac{1}{\pi}\dfrac{1}{\sqrt{x(1-x)}}$, $x \in [0,1]$, for logistic-4 function $f_a(x) = 4x(1-x)$.

Even though $\rho(x) \to \infty$ as $x \to 0^+$ and $x \to 1^-$, it is integrable:

$$\int_0^1 \rho(x)\,dx = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = 1\,. \tag{57}$$

The fact that the function $\rho(x)$ satisfies the conservation equation in (53) can be verified after a generous amount of Calculus, starting with the result (left as an exercise) that for $0 \le a \le b \le 1$,

$$\begin{aligned}\int_a^b \rho(x)\,dx &= \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{x(1-x)}}\,dx \\ &= \frac{1}{\pi}\left[\mathrm{Sin}^{-1}(2b-1) - \mathrm{Sin}^{-1}(2a-1)\right]\,. \end{aligned} \tag{58}$$

In the special case $a = 0$ and $b = 1$, the above result yields,

$$\begin{aligned}\int_0^1 \rho(x)\,dx &= \frac{1}{\pi}\left[\mathrm{Sin}^{-1}(1) - \mathrm{Sin}^{-1}(-1)\right] \\ &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right] \\ &= 1\,. \end{aligned} \tag{59}$$

This has been a very short introduction to the subject of dynamical systems and invariant measures – very little could be done which, of course, means that much has been omitted. But it was intended to be a starting point for anyone who is interested in pursuing the subject further.

One final note: The existence of the density function $\rho(x)$ in Eq. (40) is not always guaranteed. But an invariant measure μ generally exists. The complication is that the measure μ is a **measure**, and measures can be quite “irregular”. They can include things such as “Dirac delta functions,” i.e., “point masses”, which cannot be modelled with “normal functions”. This will certainly be the case when we study measures on fractal sets – at least lightly.

Lecture 20

Chaotic dynamics (cont'd)

The "Ergodic Theorem"

It is worthwhile to mention another very important idea arising from dynamical systems theory, the so-called "Ergodic Theorem." The Ergodic Theorem has been of special importance in mathematical physics with respect to the idea that

$$\textbf{"Time average"} = \textbf{"Space average"}. \tag{60}$$

Here we simply state the basic idea.

Let $I = [a, b]$ and $f : I \to I$ such that associated with f is an invariant density function $\rho(x)$ which satisfies the condition in Eq. (40) from the previous lecture (Lecture 20, Week 7), which we reproduce here: For all $S \subset I$,

$$\int_S \rho(x)\, dx = \int_{f^{-1}(S)} \rho(x)\, dx\,. \tag{61}$$

Given a suitable seed point $x_0 \in I$ (i.e., an x_0 that is not preperiodic), define

$$x_{n+1} = f(x_n)\,, \qquad n \geq 0\,. \tag{62}$$

In other words, compute the forward orbit $O(x_0)$ of x_0.

Now suppose that $g : I \to \mathbb{R}$ is a continuous function and that we evaluate g at the iterates x_n, forming the following average: For $N > 1$,

$$S_N = \frac{1}{N} \sum_{n=1}^{N} g(x_n)\,. \tag{63}$$

Then (subject to some other technical considerations which we'll ignore here), the limit of the above average exists, i.e.,

$$\lim_{N \to \infty} S_N = S\,. \tag{64}$$

Furthermore, the limit S is related to the density function ρ as follows,

$$S = \int_a^b g(x)\rho(x)\, dx\,. \tag{65}$$

Important points:

1. The quantities S_N in (63) are **time averages** – they are the average values of $g(x)$ evaluated at the iterates x_n which range over I. We consider the orbit $O(x_0) = \{x_n\}_{n=0}^N$ to define a trajectory over discrete time units, $n = 0, 1, 2, \cdots$, with $g(x)$ being evaluated at the points in I visited by the iterates.

2. The limit S in (64) is the **limiting time average**.

3. The integral in (65) is a **spatial average**. It is a weighted average of the function $g(x)$ – the weighting is performed by the invariant density function $\rho(x)$. A region $K \subset I$ that is visited more frequently by the iterates will have higher values of $\rho(x)$. This is also reflected in the time average – if the region $K \subset I$ has a higher $\rho(x)$ values, more iterates will be visiting it, which will influence the average S_N in (64).

4. Eq. (65) is the mathematical equivalent of the statement in (60).

Example 1: Consider the tent map $T(x)$ on $[0, 1]$ with invariant density function $\rho(x) = 1$. In what follows, we consider a few $g(x)$ and compute their time averages in Eq. (63) for two values of N, namely $N_1 = 10^6$, $N_2 = 10^8$. These results are compared with the corresponding spatial averages in (65). Here, since $\rho(x) = 1$,

$$S = \int_0^1 g(x)\,\rho(x)\,dx = \int_0^1 g(x)\,dx\,. \tag{66}$$

1. $g(x) = x$. We find that

$$S_{N_1} = 0.4998512593 \qquad S_{N_2} = 0.4999952251\,. \tag{67}$$

   The time averages are seen to be moving toward the limiting value

$$S = \int_0^1 x\,dx = \frac{1}{2}\,. \tag{68}$$

2. $g(x) = x^2$. We find that

$$S_{N_1} = 0.3332341836 \qquad S_{N_2} = 0.3333301506\,. \tag{69}$$

   The time averages are seen to be moving toward the limiting value

$$S = \int_0^1 x^2\,dx = \frac{1}{3}\,. \tag{70}$$

3. $g(x) = e^x$. We find that

$$S_{N_1} = 1.7180724229 \qquad S_{N_2} = 1.7182755248\,. \tag{71}$$

The time averages are seen to be moving toward the limiting value (to ten digits)

$$S = \int_0^1 e^x\,dx = e - 1 = 1.7182818285\,. \tag{72}$$

Example 2: Now consider the logistic-4 map $f_4(x) = 4x(1-x)$ with invariant density function,

$$\rho(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}\,. \tag{73}$$

Integrals involving the above density function are generally difficult to compute analytically, which provides a good motivation to approximate them using time averages. Here we shall consider only one example: A function that was quite important in an earlier lecture. For the moment, consider $a \in [0,4]$ in general and define the function,

$$g_a(x) = \ln|f_a'(x)| = \ln|a - 2ax|\,. \tag{74}$$

Recall (Lectures 13 and 14, Week 5) that the limiting time average of this function, i.e.,

$$S = \lim_{N\to\infty} \sum_{n=1}^{N} \ln|f_a'(x)|\,, \tag{75}$$

is the **Lyapunov exponent** $\lambda(x_0)$ of $f_a(x)$. As was discussed earlier, the Lyapunov exponent is computed as a time average of values of the function $\ln|f_a'(x)|$ over the iterates x_n. What we shall see here, is that the Lyapunov exponent can also be computed as a spatial average over the invariant density function $\rho(x)$.

For $g_a(x)$ in (74) with $a = 4$, i.e.,

$$g_4(x) = \ln|4 - 8x|\,, \tag{76}$$

we obtain the following numerical results, once again for $N_1 = 10^6$ and $N_2 = 10^8$,

$$S_{N_1} = 0.6931468887 \qquad S_{N_2} = 0.6931471771\,. \tag{77}$$

Note that both of these results are positive, in accord with the fact that the map $f_4(x)$ is chaotic. As such, we expect a positive Lyapunov exponent.

As $N \to \infty$, these time averages should converge to the spatial average of $g_4(x)$, namely,

$$S = \frac{1}{\pi} \int_0^1 \frac{\ln|4-8x|}{\sqrt{x(1-x)}}\, dx\,. \tag{78}$$

With a little Calculus, this integral can be computed analytically,

$$\begin{aligned} S &= \frac{1}{\pi} \int_0^1 \frac{\ln|4-8x|}{\sqrt{x(1-x)}}\, dx \\ &= \ln 2 \\ &\approx 0.6931471806\,. \end{aligned} \tag{79}$$

We see that the estimate S_{N_2} agrees with the theoretical value to 1 part in 10^{-8}.

Note: It is interesting that the Lyapunov exponent for the logistic-4 map, $\lambda = \ln 2$, is equal to that of the Tent Map.

A final note: Recall that the Lyapunov exponent for the logistic function $f_a(x)$ was defined for all $a \in [0,1]$ (the value $\lambda = -\infty$ was also acceptable). For many other a values, especially $0 < a \leq 3$, the map f_a is **not** chaotic: Orbits are attracted to fixed points or N-cycles. Clearly, we computed the Lyapunov exponents for these cases as time averages of the function $g_a(x)$ over the orbits. The question remains, are these time averages equal to spatial averages over some kind of invariant density functions or measures? The answer is "Yes." The problem is that the density functions are no longer functions – they are **generalized functions**, which include the idea of **Dirac delta functions**. For example, recall that for $0 < a < 3$, $f_a(x)$ has an attractive fixed point at $\bar{x}_2 = \dfrac{a-1}{a}$. As such, all iterates x_n will approach $\bar{x}_2$ as $n \to \infty$. In this case, the invariant measure is "$\delta(x - \bar{x}_2)$", the (unit) Dirac delta function centered at $x = \bar{x}_2$. We may discuss this later in the course.

Another final note: The idea of performing an integration of a function $f(x)$ over an interval or region of R^n by sampling over points in the region is the basis of so-called **Monte Carlo methods**. Here is a simple example. Suppose that you wish to obtain estimates of the following integral,

$$\int_a^b f(x)\, dx\,, \tag{80}$$

where the function $f(x)$ is sufficiently "nice," e.g., continuous or piecewise continuous. The first step is to design an algorithm to generate random (well, pseudo-random) numbers x_n distributed uniformly over the interval $[a, b]$. (Usually, one can find a routine that generates random numbers with a uniform distribution over $[0, 1]$. You can then scale these numbers to the interval $[a, b]$.

Now, for a large positive integer N, generate the random numbers $x_n \in [a, b]$, $1 \leq n \leq N$. At each step, compute $f(x_n)$. (You are essentially "sampling" the function at each $x_n \in [a, b]$. Then compute the average value of these sampled values,

$$A_N = \frac{1}{N} \sum_{n=1}^{N} f(x_n) \,. \tag{81}$$

As N increases, the average values A_N should provide better approximations to the integral in (80).

This might seem like an academic exercise, but the method is quite powerful in problems involving high dimensions – even, for example, in the estimation of regions in $\mathbb{R}^n$ with complicated boundaries. For a simple example, go to the following Wikipedia site,

`https://en.wikipedia.org/wiki/Monte_Carlo_method`

and scroll down to the video which illustrates the use of the Monte Carlo method to approximate the value of π. In this case, the function $f(x) = 1$ and the region of integration $D \in \mathbb{R}^2$ is the circular region of radius 1 in the first quadrant.

Here is the idea: You generate random ordered pairs (x_n, y_n) in the unit square, i.e., $0 \leq x_n \leq 1$ and $0 \leq y_n \leq 1$, distributed uniformly over the square region $[0, 1]^2 \in \mathbb{R}^2$, for $1 \leq n \leq N$. For each $n \geq 1$, evaluate $f(x_n, y_n)$, where

$$f(x, y) = \begin{cases} 1\,, & x^2 + y^2 \leq 1\,, \\ 0\,, & x^2 + y^2 > 1\,. \end{cases} \tag{82}$$

Once again, compute the average value

$$A_N = \frac{1}{N} \sum_{n=1}^{N} f(x_n, y_n) \,. \tag{83}$$

As N increases, the A_N provide better estimates to the integral

$$\int \int_D f(x, y)\, dx = \frac{\pi}{4} \,, \tag{84}$$

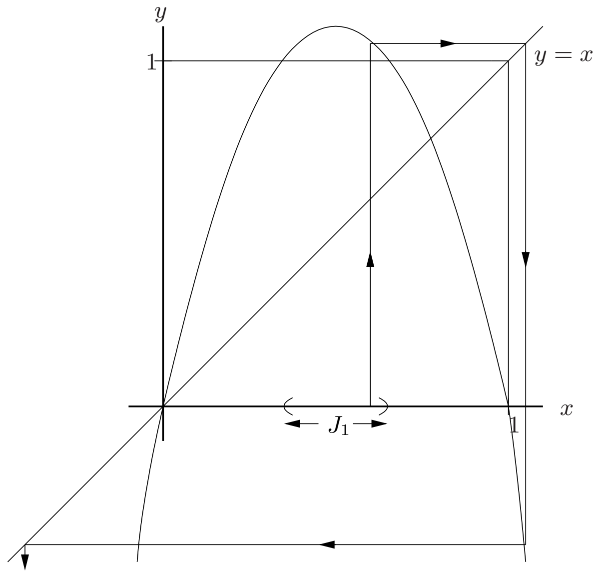
where

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\,, 0 \leq x \leq 1\,, 0 \leq y \leq 1\} \tag{85}$$

The average value A_N is essentially computing the fraction of iterates (x_n, y_n) which land in the region D.

A return to the dynamics of the Logistic Map: The case $a > 4$

We now examine the interesting iteration dynamics associated with the logistic map when $a > 4$. The graph of $f_a(x) = ax(1-x)$ for $a \cong 4.5$ is sketched below:



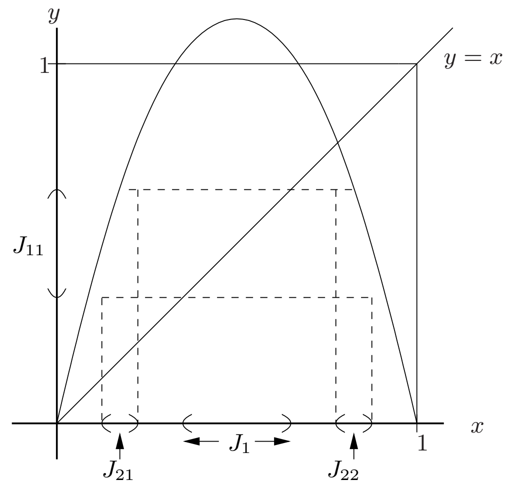
The most important qualitative feature of this graph is that f_a no longer maps $[0,1]$ into itself: There is an open interval of points, J_1, that are mapped outside the interval $[0,1]$. For $x \in J_1$, $f(x) > 1$. Note what happens to points $x \in J_1$ under further iteration of f, as shown in the figure. We note from the graph that: 1) $f^2(x) < 0$, 2) $f^{n+1}(x)$ for $n \geq 2$ and finally 3) $f^n(x) \to -\infty$ as $n \to \infty$.

The natural question that follows is "Are there any other points in $[0,1]$ that eventually leave $[0,1]$?" The answer is "Yes". Note that every point $x \in [0,1]$ has two preimages $y_1 \neq y_2$ such that $f(y_1) = f(y_2) = x$. This leads us to consider the set J_2 of all preimages of the set J_1, i.e.

$$J_2 = \{x \in [0,1] \mid f(x) \in J_1\} \tag{86}$$

Since $x \in J_2$ implies that $f(x) \in J_1$, this implies in turn that $f^2(x) \notin [0,1]$ and that $f^n(x) \to -\infty$ as $n \to \infty$. Now, what is the set J_2, the set of preimages of J_1?

We may determine the set J_2 graphically by "copying" the set J_1 onto the y-axis and then applying f "backwards", i.e. using the line $y = x$ to determine which points on the x-axis are mapped to J_1. This is shown below:



Thus, $f(J_{21}) = J_1$ and $f(J_{22}) = J_1$. The set $J_2 = J_{21} \cup J_{22}$ is the set of all points $x \in [0,1]$ such that $f^2(x) \notin [0,1]$, i.e. the set of points that leave $[0,1]$ after two applications of f.

The reader should see the pattern now. We define the set J_3 of all preimages of J_2, i.e.

$$J_3 = \{x \in [0,1] \mid f(x) \in J_2\}. \tag{87}$$

Note that $x \in J_3$ implies that $f^2(x) \in J_1$ which in turn implies that $f^3 \notin [0,1]$. Now continue this process, if possible, to define the following sets of points,

$$J_n = \{x \in [0,1] \mid f^n(x) \notin [0,1]\}, \; n \geq 1. \tag{88}$$

It is now convenient to define the following sets:

$$C_1 = [0,1] - J_1 \tag{89}$$

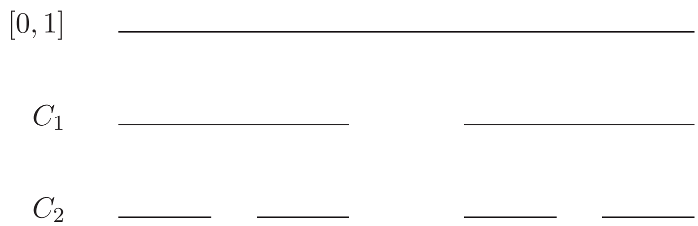
$$C_2 = C_1 - J_2 = [0,1] - (J_1 \cup J_2) \tag{90}$$

$$\vdots$$

$$C_n = C_{n-1} - J_n = [0,1] - \left(\bigcup_{k=1}^{n} J_k \right). \tag{91}$$

From the definition of the J_k, we see that for $n \geq 1$, C_n is the subset of points in $[0,1]$ that remain in $[0,1]$ after n iterations of f. The natural question is, "Is there a set of points J that remain on $[0,1]$ after any number of iterations of f?" In order to answer this, let us examine the sets C_1, C_2, etc..

From the figures shown earlier, and the definitions in (91), the sets C_1 and C_2 have the following structure:



Note that C_1 is obtained by means of a "dissection procedure" – a removal of the open set J_1 from $[0,1]$. C_2 is obtained from a "dissection" of C_1 – a removal of an open set from each of the subintervals making up C_1. Note that $[0,1]$, C_1 and C_2 are **closed intervals**. The construction of intervals C_1 and C_2 by means of a "dissection" procedure is reminiscent of the "middle-thirds dissection" procedure that was used to construct the ternary Cantor set that we discussed earlier in this course.

For the moment we simply state, without proof, the following result:

> The sequence of sets $C_1, C_2, \ldots$ converges, in the limit $n \to \infty$, to a "Cantor-like set" $C \subset [0,1]$, i.e. $\lim_{n\to\infty} C_n = C$. For any $x \in C$, $f^n(x) \in J$ for all $n \geq 0$. (The term "Cantor-like" will be defined shortly.)

In other words, **the set of points C that remain in $[0,1]$ after any number of iterations of f_a is a "Cantor-like set"**. Note that the structure of this set, i.e. the positions of points $x \in C$, $x \notin \{0,1\}$, is dependent upon the logistic map parameter a. For example, the size/length of the open interval J_1 removed form $[0,1]$ to produce J_1 is dependent upon a: As $a \to 4^+$, this interval is smaller. Of course, at $a = 4$, no dissection takes place. The reader is encouraged to find end-points of the intervals that make up the set J_1, as functions of the parameter a, hence the length of the removed interval. This size will also determine the sizes of the sets J_{21} and J_{22} removed in the next dissection procedure, although not in a linear way since the map $f_a(x)$ is not a linear function.

The reader is invited to consider an alternate definition of the set C constructed above:

$$C = \{x \in [0,1] \mid x \text{ is a periodic point of } f_a \text{ with period } n,\, n \geq 1\}.$$

Graphical analysis should show (as indeed our previous analysis of the bifurcations of f_a did) that all periodic orbits of f_a are repulsive, as they were for the case $a = 4$.

Cantor (or "Cantor-like") sets

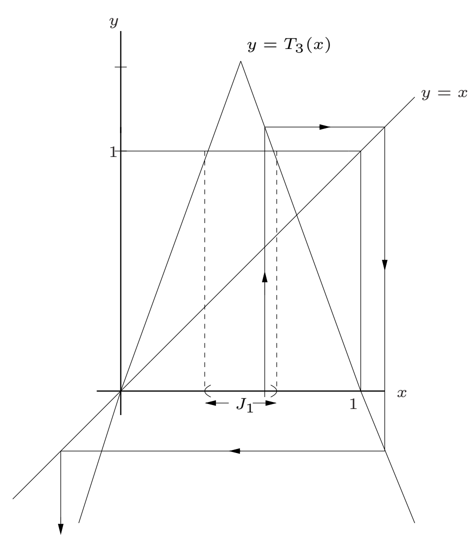
We now study Cantor-like sets in a little more detail, first by studying the famous "ternary Cantor set." We don't really have to do this by looking at the iteration of functions, since Cantor-like sets may be produced by a limiting procedure of dissection, i.e., removal of points from intervals/sets. But it is instructive to consider iteration since, after all, it is a central idea of this course.

We shall consider a family of "linearized versions" of the logistic maps $f_a(x)$, namely, the following family of modified Tent maps,

$$T_a(x) = \begin{cases} ax & 0 \leq x \leq \frac{1}{2} \\ a(1-x) & \frac{1}{2} < x \leq 1 \,. \end{cases} \tag{92}$$

(With apologies: In a previous assignment, "$T_a(x)$" was used to denote a "shifted" Tent Map.) The maximum value of $T_a(x)$ is $T_a\left(\frac{1}{2}\right) = \frac{a}{2}$. When $a = 2$, $T_2(x)$ is the Tent Map that we have studied in past lectures.

In the case $a = 3$, the graph of the Tent Map extends out of the "box", i.e., $[0,1] \times [0,1]$, as shown below. We see that T_3 maps some points in [0,1] out of [0,1]. The fate of these points, as in the case of the logistic map f_a for $a > 4$ is that under further iteration, they go to $-\infty$.



We'll proceed as we did for the logistic map f_a in the previous lecture by making a few observations:

1. Points in the (open) interval

$$J_1 = \left(\frac{1}{3}, \frac{2}{3}\right) \tag{93}$$

are mapped out of $[0,1]$ by T_3. In other words, these points leave $[0,1]$ after one application of T_3.

2. Points in $[0,1]$ that are mapped by T_2 to the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ are mapped out of $[0,1]$ after one additional application of T_3. These points lie in the **preimage** of the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ which, as in the case of the logistic map, can be determined graphically by backwards iteration to be the set

$$J_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \bigcup \left(\frac{7}{9}, \frac{8}{9}\right) . \tag{94}$$

In other words, points in J_2 leave $[0,1]$ after two applications of T_3.

This procedure can be continued, but it is perhaps easier to focus on the points in [0,1] which remain in [0,1] after a given number of iterations. As we did in the previous lecture for the logistic maps, we determine the sets of points in $[0,1]$ which remain in $[0,1]$ after n applications of the map T_3:

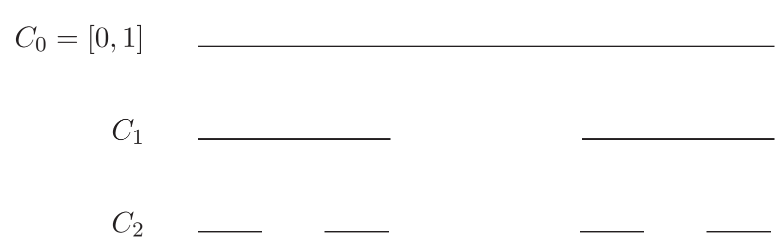
1. The set of points in $[0,1]$ which remain in $[0,1]$ after one application of T_3 comprise the set

$$C_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right] = I_{11} \cup I_{12} . \tag{95}$$

2. The set of points in $[0,1]$ which remain in $[0,1]$ after two applications of T_3 comprise the set

$$C_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right] = I_{21} \cup I_{22} \cup I_{23} \cup I_{24} . \tag{96}$$

Graphically, these intervals look as follows,



These intervals look almost identical to the intervals presented earlier for the logistic map, but there is one important difference: All of the line segments that comprise a given set C_n have equal length because the tent map $T_3(x)$ is piecewise linear.

The above results are easily extended to the general case: The set of points in $[0,1]$ which remain in $[0,1]$ after n applications of T_3 is the following set of 2^n closed intervals of length 3^{-n}:

$$C_n = \bigcup_{k=1}^{2^n} I_{nk}\,. \tag{97}$$

In other words,

$$C_n = \{x \in [0,1]\,,\ T_3^n(x) \in [0,1]\,\}\,. \tag{98}$$

From the diagram above, we see that

$$C_2 \subset C_1 \subset C_0\,. \tag{99}$$

In general,

$$C_{n+1} \subset C_n\,. \tag{100}$$

In other words, if a point x remains in [0,1] after $n+1$ iterations, which implies that $x = C_{n+1}$, then it must remain in [0,1] after only n iterations, i.e., $x = \in C_n$. But the converse does not necessarily apply: There will be points in C_n that leave after the next application of T_3.

As such, we have a nested set of **closed** intervals,

$$C_0 \supset C_1 \supset C_2 \supset \cdots\,. \tag{101}$$

We now define the set C to be the infinite intersection of this nested set, i.e.,

$$C = \bigcap_{n=0}^{\infty} C_n\,. \tag{102}$$

C is the set of points in $[0,1]$ that belong to all C_n, $n \geq 0$, i.e.,

$$C = \{x \in [0,1]\,,\ x \in C_n \text{ for all } n \geq 0\}. \tag{103}$$

Eq. (102) is stating that

$$C = \text{“} \lim_{n\to\infty} C_n \text{”}. \tag{104}$$

Note: The existence of such a **nonempty set** is guaranteed by the **Nested Intervals Theorem** which was used earlier in the course.

The set C described above is commonly known as the "ternary Cantor set" or simply "the Cantor set."

Clearly, the set C contains the points $0, 1, \frac{1}{3}$ and $\frac{2}{3}$. It contains some multiples of $\frac{1}{9}$ but not all of them, i.e., it contains the points $\frac{k}{9}$ for $k \in \{0, 1, 2, 3, 6, 7, 8, 9\}$ but not for $k \in \{4, 5\}$. In fact, it should not be difficult to see that for each $n \geq 1$, the endpoints of each of the 2^n closed intervals which comprise the set C_n are contained in C. Since each of the closed intervals has 2 endpoints, there are $2 \times 2^n = 2^{n+1}$ such points which must be contained in C. Since this is true for all $n > 1$, it follows that the number of points in C is arbitrarily large, i.e., without bound, i.e., infinite. Later, we'll show that the number of points in C is actually "beyond infinite."

And later, we shall also discover that C also contains some points that you may not have guessed, e.g., $\frac{1}{4}$ and $\frac{3}{4}$.

Here is a figure showing seven stages of the dissection process in order to show how "thin" the Cantor set C is.

Indeed, the Cantor set C has to be "thin." Recall that it lies in the intersection of all of the sets C_n which were produced by dissection. And recall that each set C_n is composed of 2^n intervals, each of length 3^{-n}. Therefore, the "length" of the set C_n is

$$L_n = (2^n)(3^{-n}) = \left(\frac{2}{3}\right)^n. \tag{105}$$

This implies that the total "length" of the Cantor set C is

$$L = \lim_{n\to\infty} L_n = 0. \tag{106}$$

It has zero length! We'll return to this idea a little later.

The Cantor set C, and all Cantor-like sets (we'll define this shortly), is a fascinating set. Here, we shall prove a few basic, but very interesting properties. We'll show that C is

1. bounded,

2. closed,

3. totally disconnected,

4. perfect and

5. uncountable.

Definition: A set of real numbers S that has properties 1-4 is said to be a **Cantor-like set** or, simply a **Cantor set**. (The book by Gulick simply calls these sets "Cantor sets", but many references use the term "Cantor-like", the terminology that will be employed, for the most part, in this course.)

We now prove the properties listed above.

1. The set C is bounded.

The formal definition of a bounded set is as follows:

Definition: A set $S \subset \mathbb{R}$ is bounded if there exists an $M \geq 0$ such that

$$|x| \leq M \quad \text{for all} \quad x \in S\,.$$

Proof that the set C is bounded: Rather trivial. Since $C \subset [0, 1]$, it follows that if $x \in C$, then $0 \leq x \leq 1$ which, in turn, implies that

$$|x| \leq 1\,. \tag{107}$$

Therefore C is bounded.

To be continued ...

Lecture 21

Cantor and Cantor-like sets (cont'd)

We continue with the proofs of several properties of the Cantor set.

2. The set C is closed.

Definition: A set S is **closed** if it contains all of its limit points: If $\{x_n\} \in S$ and $\lim_{n\to\infty} x_n = x$ then $x \in S$.

Examples:

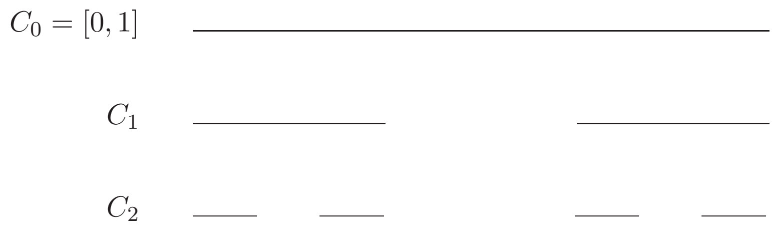
1. The set of real numbers $\mathbb{R}$ is closed. The limit x of any convergent sequence $\{x_n\} \in \mathbb{R}$ is a real number. (The fact that the set $\mathbb{R}$ is unbounded might bother some people, because of some other ideas from analysis. But closed sets do not have to be bounded.)

2. The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is not closed. It is possible to have a convergent set of rational numbers $x_n \in \mathbb{Q}$ that converges to an irrational number $x \notin \mathbb{Q}$. For example, the sequence of rational numbers,
$$x_1 = 3\,, \ x_2 = \frac{31}{10}\,, \ x_3 = \frac{314}{100}\,, \ x_4 = \frac{3141}{1000} \cdots ,$$
converges to the irrational number $\pi \notin \mathbb{Q}$.

3. The set $[0, 1]$ is closed.

4. The set $(0, 1]$ is not closed. All points of the sequence, $x_n = \dfrac{1}{n}$, $n \geq 1$, belong to $(0, 1]$. And this sequence is convergent, since $\lim_{n\to\infty} \dfrac{1}{n} = 0$. But the limit of this sequence does not lie in $(0, 1]$. (Note that $(0, 1]$ is not closed, but that does not imply that it is open. That's another definition.)

Before proving that C is closed, let us recall the "middle-thirds dissection" procedure involved in its construction. We started with the set/interval $C_0 = [0, 1]$ and removed the middle-third open set $(\frac{1}{3}, \frac{2}{3})$ to produce the set C_1. We then removed the middle-third open set of C_1 to produce C_2 and so on.



At the nth stage, we have the set C_n which is a union of 2^n intervals of length 3^{-n}. The net result is a nested set of **closed sets**,

$$C_0 \supset C_1 \supset C_2 \supset \cdots . \tag{108}$$

The Cantor set C is defined to be the infinite intersection of this nested set, i.e.,

$$C = \bigcap_{n=0}^{\infty} C_n . \tag{109}$$

As a result, C is the set of points in [0,1] that belong to all sets C_n, $n \geq 0$. We now proceed with the proof.

Proof that C is closed: Let $A \in C$ be a convergent sequence of points in C, i.e., $A = \{x_n\}_{n=1}^{\infty} \subset C$ such that $\lim_{n\to\infty} x_n = x$. We now prove that $x \in C$.

Recall that the Cantor set is contained in each set C_n for $n \geq 0$. Since A is a subset of C, it follows that $A \subset C_n$ for all $n \geq 0$, i.e., all points in A belong to each set C_n. But each C_n is a finite union of closed intervals, which implies that each C_n is a closed set, i.e., it contains all of its limit points.

Note: Here we simply remark, without proof, that a finite union of closed sets must be closed. An infinite union of closed sets does not have to be closed. The proofs of these statments would be the subject of a course dedicated to Real Analysis.

For each $n \geq 0$, $A \subset C_n$ which implies that

$$\lim_{n\to\infty} x_n = x \in C_n .$$

This implies that $x \in C_n$ for all $n \geq 0$. But from the definition of C, i.e., Eq. (103), this implies that $x \in C$. Therefore C is closed (i.e., it contains its limit points.

Here is another look at the dissection procedure which produces the Cantor set C. You might ask the question, " The set C is very thin and has many, many gaps. How can there even exist a sequence of points $x_n \in C$ which converges to a point $x \in C$?"

Here is an example: The set of points

$$A = \left\{ \frac{1}{3^n} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \cdots \right\} . \tag{110}$$

All of the points $x_n = \dfrac{1}{3^n}$, $n \geq 1$, belong to the set C. The limit of this sequence is $x = 0$, which is also an element of C.

3. The set C is totally disconnected.

In order to talk about a set being "totally disconnected," we should first discuss the idea of a set being "disconnected." A set $S \subset \mathbb{R}$ (and more generally, in $\mathbb{R}^n$) is disconnected if it is "not connected," i.e., "not in one piece." For example, the set

$$S_1 = [0,1] \cup [2,3]$$

is not connected. In order to get from one "piece" of S, say $[0,1]$, to the other, $[2,3]$, you have to leave the set S, i.e., enter the territory $(1,2) \notin S$. Even the set

$$S_2 = [0,1) \cup (1,2] \, ,$$

which is obtained from the set $[0,2]$ by removing the single point 1, is not connected. In order to get from the piece $[0,1)$ to the other piece $(1,2]$, you have to leave the set S. This might seem a little strange since you can find sets of points, one from [0,1) and the other from (1,2] which are arbitrarily close to each other. But the point 1 is missing, and that's that.

There is a mathematical way to define disconnectedness. It will have to involve the concept of **open sets**, which we haven't yet defined formally. The following definitions are formulated over the real line $\mathbb{R}$ and can be generalized to arbitrary **metric spaces**.

Definition: The **open interval** (or, in general, "**open ball**") of radius r about a point $p \in \mathbb{R}$ is the set

$$B_r(p) = \{x \in \mathbb{R} \mid |x - p| < r\} = (p - r, p + r) .$$

It is an interval (or ball) of radius r centered at p.

Definition: A set $S \subset \mathbb{R}$ is **open** if for each $p \in S$, there exists an $r > 0$ so that $B_r(p) \subset S$.

Examples:

1. The set $S = (0, 1)$ is open. No matter how close a point p is to one of the endpoints of S, say 0, if we choose $r = p/2$, then the open ball $B_r(p)$ lies entirely in S. In fact, for any $r < p$, the open ball $B_r(p)$ lies entirely in S.

2. The set $S = [0, 1]$ is not open. If we choose $p = 0$, then there there exists no $r > 0$ for which the open ball $B_r(0)$ lies entirely in S.

3. The set $S = (0, 1]$ is not open. If we choose $p = 1$, then there is no open ball $B_r(1)$ which lies entirely in S.

Note: In Example 2, $S = [0, 1]$, which is not open, is closed. But in Example 3, $S = (0, 1]$, which is not open, is not closed. This is to illustrate the fact that, formally, "open" and "closed" are **not** opposites or negations of each other. If a set S is not one of the two, it is not necessarily the other of the two.

Definition: A set $S \in \mathbb{R}$ is disconnected if there exist two disjoint open sets $A, B \subset \mathbb{R}$, i.e., $A \cap B = \phi$ (where ϕ denotes the null set) such that $A \cap S \neq \phi$, $B \cap S \neq \phi$ and

$$S \in A \cup B .$$

In other words, the set S is contained in two disjoint open sets A and B, each of which contain at least one point of S.

For the set S_1 defined earlier,

$$S_1 = [0,1] \cup [2,3]\,,$$

we can use the disjoint open intervals,

$$A = \left(-1, \frac{5}{4}\right) \quad B = \left(\frac{3}{2}, 4\right)\,.$$

For the set S_2 defined earlier,

$$S_2 = [0,1) \cup (1,2]\,,$$

we can use the open sets

$$A = (-1,1) \quad B = (1,3)\,.$$

Note that there is not much flexibility for A and B when it comes to the point 1. We must be able to include the first piece $[0,1)$ in A and the second piece $(1,2]$ in B, with A and B not intersecting. This would not be possible if we demanded that the sets A and B were closed, since they would have to intersect at 1.

Now what about "total disconnectedness" of a set S? Loosely, it means that one cannot get from one point $x \in S$ to another point $y \in S$ without leaving the set S. In a sense, it means that $S \subset \mathbb{R}$ is composed of points, but has no intervals of the form (a,b) in it. We can state this condition mathematically in terms of open sets:

Definition: A set $S \in \mathbb{R}$ is totally disconnected if, for any distinct $x, y \in S$, there exists a pair of disjoint, open sets A and B such that

$$x \in A\,, \quad y \in B \quad \text{and} \quad S \in A \cup B\,.$$

(Note that A and B are not fixed, i.e., they will depend on x and y.)

Examples:

1. Any finite set of distinct points x_i, $1 \leq i \leq N$, is not only disconnected, but totally disconnected.
2. An infinite set of distinct points x_i is also disconnected, for example, $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. And even the set $\{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is totally disconnected.

3. The the set of rational numbers $\mathbb{Q}$ is totally disconnected (even though it is dense on $\mathbb{R}$).

Proof that the Cantor set C is totally disconnected: We shall use the fact that if C is not totally disconnected, it must contain at least one interval $(a, b) \in C$, where $a < b$. (In other words, you'll be able to move from $a \in C$ to $b \in C$ without leaving C.) We shall also use the fact that the length of an interval (a, b) is $b - a > 0$.

Now recall the fact that the set $C \in C_n$ for all $n \geq 0$. Also recall the fact that for a given $n \geq 0$, C is the union of 2^n closed intervals I_{nk} of length $\frac{1}{3^n}$. Then, for a given $n \geq 0$, if the interval $(a, b) \in C$ and $C \in C_n$, it means that the length, $L = b - a$, of the interval (a, b) must be less than $\frac{1}{3^n}$, i.e.,

$$b - a < \frac{1}{3^n}\,. \tag{111}$$

But the above inequality must be true for all $n \geq 0$, i.e., as $n \to \infty$. For the inequality to be true for all $n \geq 0$, we must have $a = b$, implying that no intervals of the form (a, b), with $a < b$, exist in C. Therefore C is totally disconnected.

4. The set C is perfect.

Of course, we'll need a definition of "perfect sets."

Definition: A set $S \in \mathbb{R}$ (or, in general $\mathbb{R}^n$) is **perfect** if every point $x \in S$ is the limit point of a sequence of other points in S. In other words, for any point $x \in S$, there exists a sequence $\{y_n\} \subset S$, with $y_n \neq x$ for all n, such that $\lim_{n\to\infty} y_n = x$.

Examples:

1. A finite set of distinct points $\{x_n\}_{n=1}^{N}$ is not perfect.
2. The set of real numbers $\mathbb{R}$ is perfect.
3. The interval $[0, 1]$ is perfect.
4. The interval $(0, 1)$ is perfect.
5. The set $S = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is not perfect.

6. The set $S = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is not perfect.

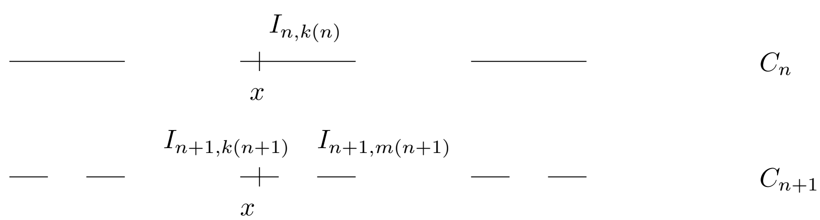
Before we continue with the proof, the idea that the Cantor set C is perfect may seem quite strange. After all, the Cantor set C has a lot of "holes" in it. How could **any** point $x \in C$ be the limit of a sequence of other points $y_n \in C$? We'll see....

Proof: Let $x \in C$. Recall that by definition, $x \in C_n$ for all $n \geq 0$ where each set C_n is a union of 2^n closed intervals of length $\frac{1}{3^n}$, i.e.,

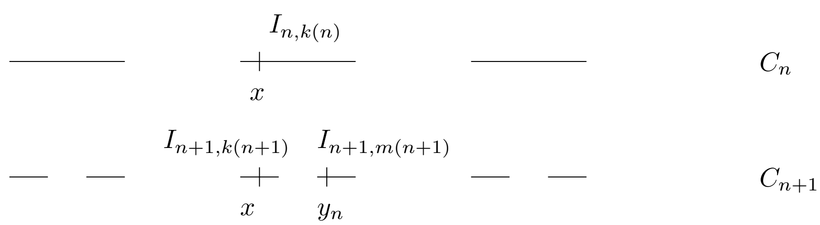
$$C_n = I_{n1} \cup I_{n2} \cup \cdots \cup I_{n2^n} \,. \tag{112}$$

This means that x lies in one of the segments I_{nk}. We'll identify this particular segment as $I_{n,k(n)}$, to indicate that there is a specific index $k(n) \in \{1, 2, \cdots, 2^n\}$ associated with x for each n.

Let's now consider the fact that $x \in C_{n+1}$, i.e., x lies in one of the 2^{n+1} segments $I_{n+1,k}$. Following the earlier procedure, we'll call this segment $I_{n+1,k(n+1)}$. The interval $I_{n+1,k(n+1)}$ containing x was produced by a dissection of interval $I_{n,k(n)}$ as sketched below. There are two possibilities, depending on whether x lies on the left third of $I_{n,k(n)}$ or the right third. The former case is sketched below.



Whatever the situation, we pick a point $y_n \in C$ that belongs to the other subinterval $I_{n+1,k}$ that came from $I_{n,k(n)}$ but that **does not** contain x – we'll identify that interval as $I_{n+1,m(n+1)}$. With respect to the case sketched above, y_n would be selected as below.



Since both x and y_n are contained in $I_{n,k(n)}$, it follows that

$$|x - y_n| \leq \frac{1}{3^n} \,. \tag{113}$$

But since x and y_n belong to **different** subintervals $I_{n+1,k}$, $y_n \neq x$.

We now repeat the procedure by replacing n with $n+1$ above, i.e., employing the fact that $x \in C_{n+2}$. It will lie in an interval $I_{n+2,k(n+2)}$ which was produced by a dissection of interval $I_{n+1,k(n+1)}$. We'll then choose a point $y_{n+1} \in C$ that belongs to the other subinterval $I_{n+2,k}$ that came from $I_{n+1,k(n+1)}$ but that **does not** contain x – we'll identify this interval as $I_{n+2,m(n+2)}$. Since both x and y_{n+1} are contained in $I_{n+1,k(n+1)}$, it follows that

$$|x - y_{n+1}| \leq \frac{1}{3^{n+1}} . \tag{114}$$

And once again, since x and y_{n+1} belong to different subintervals $I_{n+1,k}$, $y_{n+1} \neq x$. We now continue this procedure indefinitely, letting $n \to \infty$ to produce a set of points $\{y_n\}$ distinct from x but such that

$$|x - y_n| \to 0 \quad \text{as} \quad n \to \infty . \tag{115}$$

In other words,

$$\lim_{n \to \infty} y_n = x , \tag{116}$$

where all points $y_n \neq x$. Therefore, $x \in C$ is a limit point of a sequence of points $\{y_n\} \in C$ which are all different from x. Since this is true for all $x \in C$, the set C is perfect.