Iterated function systems on multifunctions

D. La Torre, F. Mendivil and E.R. Vrscay

[1]Department of Economics, Business and Statistics,
University of Milan, Italy

[2]Department of Mathematics and Statistics,
Acadia University, Wolfville, Nova Scotia, Canada

[3]Department of Applied Mathematics,
University of Waterloo, Waterloo, Ontario, Canada

Abstract. We introduce a method of iterated function systems (IFS) over the space of set-valued mappings (multifunctions). This is done by first considering a couple of useful metrics over the space of multifunctions $\mathcal{F}(X,Y)$. Some appropriate IFS-type fractal transform operators $T : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ are then defined which combine spatially-contracted and range-modified copies of a multifunction u to produce a new multifunction $v = Tu$. Under suitable conditions, the fractal transform T is contractive, implying the existence of a fixed-point set-valued mapping $\bar{u}$. Some simple examples are then presented.
We then consider the inverse problem of approximation of set-valued mappings by fixed points of fractal transform operators T and present some preliminary results.

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Dedicated to the memory of Professor Bruno Forte

1 Introduction

In this paper, we introduce a method of iterated function systems (IFS) over spaces of set-valued mappings or *multifunctions*. The idea of studying the action of sets of contraction mappings in $\mathbb{R}^n$ can be traced back to a number of very interesting historical papers. However, the landmark papers by Hutchinson [8] and Barnsley and Demko [2] showed how such systems of contractive maps with associated probabilities – called "iterated function systems" by the latter – acting in a parallel manner, either deterministically or probabilistically, could be used to construct fractal sets and measures.

This formulation of an IFS-type method over multifunction represents recent results of an ongoing research programme on the construction of appropriate IFS-type operators, or *generalized fractal transforms*, over various spaces, i.e., function spaces and distributions [6,7], vector-valued measures [13], integral transforms [5] and wavelet transforms [12,15]. Very briefly, and at the risk of sacrificing rigor, the action of a GFT T on an element u of the complete metric space (X, d) under consideration can be summarized as follows: (i) it produces a set of N spatially-contracted copies of u, (ii) it then modifies the values of these copies by means of a suitable range-mapping and finally (iii) it recombines these copies using an appropriate operator to produce the element $v \in X$, $v = Tu$. (In the case of fractal-wavelet transforms [12,15], the copies of u in (i) are actually subtrees of a tree that are then copied onto lower positions of the tree.)

In each of the above-mentioned cases, the fractal transform T is guaranteed to be contractive when the parameters defining it satisfy appropriate conditions specific to the metric space of concern. In this situation, Banach's fixed point theorem guarantees the existence of a unique fixed point $\bar{u} = T\bar{u}$.

The *inverse problem* of fractal-based approximation is as follows: Given an element y, can we find a fractal transform T with fixed point $\bar{u}$ so that $d(y, \bar{u})$ is sufficiently small. However, the search for such transforms is enormously complicated. Thanks to a simple consequence of Banach's fixed point theorem known as the "Collage Theorem" (to be discussed below), most practical methods of solving the inverse problem seek to find an operator T for which the *collage distance* $d(u, Tu)$ is as small as possible.

In this paper, as stated above, we formulate some IFS-type fractal transform operators on the space of set-valued mappings over closed and bounded intervals of $\mathbb{R}^n$. We first consider a couple of metrics over these spaces and then establish the Lipschitz constants of the fractal transforms in these metrics. Some graphical examples are then presented.

Finally, we present an application of this method of "IFS over multifunctions" (IFSMF) to fractal image coding and present a simple example of an IFSMF-coded image multifunction.

2 Preliminary results on Hausdorff distance

In the following we will denote by $\mathcal{H}(Y)$ the space of all non-empty compact subsets of Y and by $d_h(A, B)$ the Hausdorff distance between A and B, that is

$$d_h(A, B) = \max\{\max_{x \in A} d(x, B), \max_{x \in B} d(x, A)\},$$

where $d(x, y)$ is the Euclidean norm and $d(x, A)$ is the usual distance between the point x and the set A, i.e.,

$$d(x, A) = \min_{y \in A} d(x, y).$$

It is well known that the space $(\mathcal{H}(Y), d_h)$ is a complete metric space if Y is complete [8]. We now prove some results concerning this metric.

We now prove two results will be crucial for proving the contractivity of IFS operators on multifunctions.

Lemma 1. *Let $A, B, I \subset \mathbb{R}^n$. Then $d_h(A+I, B+I) \le d_h(A, B)$.*

Proof. We see that

$$\begin{aligned} d(A+I, B+I) &= \max_{a+i} \min_{b+j} \|(a+i)-(b+j)\| \\ &\le \max_{a+i} \min_{b} \|(a+i)-(b+i)\| \\ &= \max_{a+i} \min_{b} \|a-b\| = d(A,B). \end{aligned}$$

By symmetry we also have $d(B+I, A+I) \le d(B, A)$, which gives the desired result.

Lemma 2. *Let $A_1, A_2, B_1, B_2 \subset \mathbb{R}^n$ and $\lambda_i \ge 0$. Then*

$$d_h(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) \le \lambda_1 d_h(A_1, B_1) + \lambda_2 d_h(A_2, B_2).$$

Proof. Computing, we see that

$$\begin{aligned} d(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) &= \max_{a_1,a_2} \min_{b_1,b_2} \|\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 b_1 - \lambda_2 b_2\| \\ &\le \max_{a_1,a_2} \min_{b_1,b_2} [\lambda_1\|a_1-b_1\| + \lambda_2\|a_2-b_2\|] \\ &= \lambda_1 \max_{a_1} \min_{b_1} \|a_1-b_1\| + \lambda_2 \max_{a_2} \min_{b_2} \|a_2-b_2\| \\ &= \lambda_1 d(A_1, B_1) + \lambda_2 d(A_2, B_2). \end{aligned}$$

Similarly we have that $d(\lambda_1 B_1 + \lambda_2 B_2, \lambda_1 A_1 + \lambda_2 A_2) \le \lambda_1 d(B_1, A_1) + \lambda_2 d(B_2, A_2)$. Since $d(A_1, B_1) \le d_h(A_1, B_1)$ and $d(B_1, A_1) \le d_h(A_1, B_1)$, we have the desired result.

Corollary 1. *Let $A_i, B_i \subset \mathbb{R}^n$ and $\lambda_i \ge 0$ for $i = 1, 2, \ldots, N$. Then*

$$d_h(\sum_i \lambda_i A_i, \lambda_i B_i) \le \sum_i \lambda_i \ d_h(A_i, B_i).$$

It is easy to see that if A is convex and $\lambda_i \ge 0$ with $\sum_i \lambda_i = 1$ then $A = \sum_i \lambda_i A$. Using this observation and the previous result we easily get the following lemma.

Lemma 3. *Let $A, B, C \subset \mathbb{R}^n$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$. Suppose that A, B, C are compact and A is convex. Then*

$$d_h(A, \lambda_1 B + \lambda_2 C) \le \lambda_1 d_h(A, B) + \lambda_2 d_h(A, C).$$

Example 1. The previous lemma is not true without the convexity of the set A; for instance, take

$$\begin{gathered} A = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, y = 1\} \cup \{(x,y) \in \mathbb{R}^2 : x = 0, 1/2 \le y \le 1\} \\ \cup \{(x,y) \in \mathbb{R}^2 : x = 1, 1/2 \le y \le 1\} \end{gathered}$$

and $B = (0,0)$, $C = (1,0)$, $\lambda_1 = \lambda_2 = 1/2$. Then

$$d_h(A, \lambda_1 B + \lambda_2 C) = 1 \ge \lambda_1 d_h(A, B) + \lambda_2 d_h(A, C) = 1/2.$$

3 Some IFS operators on multifunctions

The aim of this section is to introduce some IFS operators of the space of multifunctions. We recall that a setvalued mappings or multifunction $F : X \rightrightarrows Y$ is a function from X to the power set 2^Y. We recall that the graph of F is the following subset of $X \times Y$

$$\operatorname{graph} F = \{(x, y) \in X \times Y : y \in F(x)\}\,.$$

If $F(x)$ is a closed, compact or convex we say that F is closed, compact or convex valued, respectively. Let $(X, \mathbb{B}, \mu)$ be a finite measure space; a multifunction $F : X \to Y$ is said to be measurable if for each open $O \subset Y$ we have

$$F^{-1}(O) = \{x \in X : F(x) \cap O \neq \emptyset\} \in \mathbb{B}$$

A function $f : X \to Y$ is a selection of F if $f(x) \in F(x)$, $\forall x \in X$. In the following we will suppose that Y is compact and $F(x)$ is compact for each $x \in X$. Define

$$\mathcal{F}(X, Y) = \{F : X \to \mathcal{H}(Y)\}.$$

We place the following two metrics on $\mathcal{F}(X, Y)$; the first is

$$d_\infty(F, G) = \sup_{x \in X} d_h(F(x), G(x))$$

and the second (here μ is a finite measure on X and $p \geq 1$)

$$d_p(F, G) = \left(\int_X d_h(F(x), G(x))^p \; d\mu(x) \right)^{1/p}.$$

Proposition 1. *The space $(\mathcal{F}(X, Y), d_\infty)$ is a complete metric space.*

Proof. It is trivial to prove that $d_\infty(F, G) = 0$ if and only if $F = G$ and that $d_\infty(F, G) = d_\infty(G, F)$. Furthermore for all $F, G, L \in \mathcal{F}(X, Y)$ we have

$$\begin{aligned} d_\infty(F, G) &= \sup_{x \in X} d_h(F(x), G(x)) \\ &\leq \sup_{x \in X} d_h(F(x), L(x)) + d_h(L(x), G(x)) \\ &\leq \sup_{x \in X} d_h(F(x), L(x)) + \sup_{x \in X} d_h(L(x), G(x)) \\ &= d_\infty(F, L) + d_\infty(L, G) \end{aligned}$$

To prove that it is a complete, let F_n be a Cauchy sequence of elements of $\mathcal{F}(X, Y)$; so $\forall \epsilon > 0$ there exists $n_0(\epsilon) > 0$ such that for all $n, m \geq n_0(\epsilon)$ we have $d_\infty(F_n, F_m) \leq \epsilon$. So for all $x \in X$ and for all $n, m \geq n_0(\epsilon)$ we have $d_h(F_n(x), F_m(x)) \leq \epsilon$ and the sequence $F_n(x)$ is Cauchy in $\mathcal{H}(Y)$. Since it is complete there exists $A(x)$ such that $d_h(F_n(x), A(x)) \to 0$ when $n \to +\infty$. So for all $x \in X$ and for all $n, m \geq n_0(\epsilon)$ we have $d_h(F_n(x), F_m(x)) \leq \epsilon$ and sending $m \to +\infty$ we have $d_h(F_n(x), A(x)) \leq \epsilon$ that is $d_\infty(F_n, A) \leq \epsilon$.

Proposition 2. *d_p is a (pseudo) metric on $\mathcal{F}(X,Y)$.*

Proof. It is clear that $d_p(F,G)=0$ iff $d_h(F(x),G(x))=0$ for μ almost all $x\in X$ which happens iff $F(x)=G(x)$ for μ almost all $x\in X$. It is also clear that d_p is symmetric. For the triangle inequality, notice that

$$\begin{aligned} d_p(F,G) &= \left(\int_X d_h(F(x),G(x))^p \ d\mu(x)\right)^{1/p} \\ &\le \left(\int_X [d_h(F(x),H(x)) + d_h(H(x),G(x))]^p \ d\mu(x)\right)^{1/p} \\ &\le \left(\int_X d_h(F(x),H(x))^p \ d\mu(x)\right)^{1/p} + \left(\int_X d_h(H(x),G(x))^p \ d\mu(x)\right)^{1/p} \\ &= d_p(F,H) + d_p(H,G). \end{aligned}$$

Notice that we only get a pseudo-metric since functions which differ only on a set of μ measure zero will clearly be zero distance apart. However, this is the usual situation with the L^p spaces.

Proposition 3. *Let Y be a compact interval of $\mathbb{R}$ and suppose that $F(x)$ is convex for each $x\in X$ and for all $F\in\mathcal{F}(X,Y)$. Suppose that all $F\in\mathcal{F}(X,Y)$ are measurable. Then $\mathcal{F}(X,Y)$ is complete under d_p.*

Proof. To prove that it is a complete, let F_n be a Cauchy sequence of elements of $\mathcal{F}(X,Y)$; so $\forall\epsilon>0$ there exists $n_0(\epsilon)>0$ such that for all $n,m\ge n_0(\epsilon)$ we have $d_p(F_n,F_m)\le\epsilon$. Since $F_n(x)$ is compact and convex then $F_n(x) = [\min F_n(x), \max F_x(x)]$. The functions $\phi_n^*(x)=\min F_n(x)$ and $\phi_n^{**}(x)=\max F_n(x)$ are measurable and

$$\|\phi_n^*(x)-\phi_m^*(x)\|_p \le d_p(F_n,F_m)$$

$$\|\phi_n^{**}(x)-\phi_m^*(x)\|_p \le d_p(F_n,F_m)$$

and so ϕ_n^* and ϕ_n^{**} are Cauchy in $L^p(X)$. So there exists ϕ^* and ϕ^{**} such that $\phi_n^*\to\phi^*$ and $\phi_n^{**}\to\phi^{**}$ in the usual L^p metric. If we build the function $F(x) = [\phi^*(x),\phi^{**}(x)]$ then

$$\begin{aligned} d_p(F_n,F) &= \left(\int_X d_h(F_n(x),F(x))^p \ d\mu(x)\right)^{1/p} \\ &= \left(\int_X \max\{|\phi_n^*(x)-\phi^*(x)|^p, |\phi_n^{**}(x)-\phi^{**}(x)|^p\} \ d\mu(x)\right)^{1/p} \\ &\le \left(\int_X |\phi_n^*(x)-\phi^*(x)|^p \ d\mu(x)\right)^{1/p} \\ &\quad + \left(\int_X |\phi_n^{**}(x)-\phi^{**}(x)|^p \ d\mu(x)\}\right)^{1/p} \end{aligned}$$

Having these preliminaries out of the way, in next sections we define a two IFS-type operators on $\mathcal{F}(X,Y)$.

3.1 The union operator

Let $w_i : X \to X$ be maps on X and $\phi_i : \mathcal{H}(Y) \to \mathcal{H}(Y)$ with Lipschitz constants K_i. Define $T : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ by

$$T(F)(x) = \bigcup_i \phi_i(F(w_i^{-1}(x))).$$

Proposition 4. *If $K = \max_i K_i < 1$, then T is contractive in d_∞.*

Proof. We compute that

$$\begin{aligned}
d_\infty(T(F),T(G)) &= \sup_x \; d_h\left(\bigcup_i \phi_i(F(w_i^{-1}(x))), \bigcup_i \phi_i(G(w_i^{-1}(x)))\right) \\
&\le \sup_x \max_i \; d_h\left(\phi_i(F(w_i^{-1}(x))), \phi_i(G(w_i^{-1}(x)))\right) \\
&\le \sup_x \max_i K_i d_h\left(F(w_i^{-1}(x)), G(w_i^{-1}(x))\right) \\
&\le K \sup_z \; d_h(F(z),G(z)) = K d_\infty(F,G).
\end{aligned}$$

The result follows.

Proposition 5. *Assume that $d\mu(w_i(x)) \le s_i d\mu(x)$ where $s_i \ge 0$. Then*

$$d_p(T(F),T(G)) \le \left(\sum_i K_i^p s_i\right)^{1/p} d_p(F,G).$$

Proof. Computing, we get

$$\begin{aligned}
d_p(T(F),T(G)) &= \left\{\int_X d_h\left[\bigcup_i \phi_i(F(w_i^{-1}(x))), \bigcup_i \phi_i(G(w_i^{-1}(x)))\right]^p \; d\mu(x)\right\}^{1/p} \\
&\le \left\{\int_X \max_i d_h\left[\phi_i(F(w_i^{-1}(x))), \phi_i(G(w_i^{-1}(x)))\right]^p \; d\mu(x)\right\}^{1/p} \\
&\le \left\{\int_X \max_i K_i \; d_h\left[F(w_i^{-1}(x)), G(w_i^{-1}(x))\right]^p \; d\mu(x)\right\}^{1/p} \\
&= \left\{\sum_i K_i^p \int_{M_i} d_h\left[F(w_i^{-1}(x)), G(w_i^{-1}(x))\right]^p \; d\mu(x)\right\}^{1/p} \\
&\le \left\{\sum_i K_i^p \int_{w_i(X)} d_h\left[F(w_i^{-1}(x)), G(w_i^{-1}(x))\right]^p \; d\mu(x)\right\}^{1/p} \\
&\le \left\{\sum_i K_i^p s_i \int_X d_h\left[F(z), G(z)\right]^p \; d\mu(z)\right\}^{1/p} \\
&= \left[\sum_i K_i^p s_i\right]^{1/p} d_p(F,G).
\end{aligned}$$

In the above, we have used the sets $M_i \subset w_i(X)$ defined by

$$M_i = \left\{ x \in X : d_h(F(w_i^{-1}(x)), G(w_i^{-1}(x))) \geq d_h(F(w_j^{-1}(x)), G(w_j^{-1}(x))) \text{ for all } j \right\}.$$

That is, the set M_i consists of all those points for which the ith preimage gives the largest Hausdorff distance.

Notice that if $X \subset \mathbb{R}$ and μ is Lebesgue measure and $w_i(x)$ satisfy $|w_i'(x)| \leq s_i$ then the condition $d\mu(w_i(x)) \leq s_i d\mu(x)$ is satisfied. This is the situation that is used in image processing applications.

3.2 The sum operator

With a similar setup as in the previous section, define the operator $T : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ by

$$T(F)(x) = \sum_i p_i(x)\phi_i(F(w_i^{-1}(x)))$$

where the sum depends on x and is over those i so that $x \in w_i(X)$. We require that the functions p_i satisfy that $\sum_i p_i(x) = 1$ (again, with the dependence of the sum on x).

The idea is to average the contributions of the various components in the areas where there is overlap.

Proposition 6. *We have*

$$d_\infty(T(F), T(G)) \leq \left[\sup_x \sum_i p_i(x) K_i \right] d_\infty(F, G).$$

Proof. We compute and see that

$$\begin{aligned} d_\infty(T(F), T(G)) &= \sup_x \; d_h\left(\sum_i p_i(x)\phi_i(F(w_i^{-1}(x))), \sum_i p_i(x)\phi_i(G(w_i^{-1}(x))) \right) \\ &\leq \sup_x \sum_i p_i(x) K_i d_h(F(w_i^{-1}(x)), G(w_i^{-1}(x))) \\ &\leq \left[\sup_x \sum_i p_i(x) K_i \right] d_\infty(F, G). \end{aligned}$$

Lemma 4. *Let $a_i \in \mathbb{R}$, $i = 1 \dots n$. Then*

$$\left| \sum_i a_i \right|^p \leq C(n)^p \sum_i |a_i|^p,$$

with $C(n) = n^{(p-1)/p}$. Thus if $p = 1$, we can choose $C(n) = 1$.

Proposition 7. *Let* $p_i = \sup_x p_i(w_i(x))$ *and* $s_i \geq 0$ *be such that* $d\mu(w_i(x)) \leq s_i d\mu(x)$*. Then we have*

$$d_p(T(F), T(G)) \leq C(n) \left(\sum_i K_i^p s_i^p p_i^p \right)^{1/p} d_p(F, G).$$

Proof. We compute and see that

$$\begin{aligned} d_p(T(F), T(G))^p &= \int_X \left(d_h \left(\sum_i p_i(x) \phi_i(F(w_i^{-1}(x))), \sum_i p_i(x) \phi_i(G(w_i^{-1}(x))) \right) \right)^p \; d\mu(x) \\ &\leq \int_X \left(\sum_i p_i(x) \; K_i \; d_h \left(F(w_i^{-1}(x)), G(w_i^{-1}(x)) \right) \right)^p \; d\mu(x) \\ &\leq \int_{w_i(X)} C(n)^p \sum_i p_i(x)^p \; K_i^p \; \left(d_h \left(F(w_i^{-1}(x)), G(w_i^{-1}(x)) \right) \right)^p \; d\mu(x) \\ &\leq C(n)^p \sum_i K_i^p s_i^p \int_X p_i(w_i(z))^p \; d_h \left(F(z), G(z) \right)^p \; d\mu(z) \\ &\leq C(n)^p \left(\sum_i K_i^p s_i^p p_i^p \right) d_p(F, G)^p. \end{aligned}$$

Notice that it is easy (but messy) to tighten the estimate in the Proposition.

4 Applications to fractal image coding and the inverse problem

We now present some practical realizations and applications of IFSMF with particular focus on the coding of signals and images. The idea of this section is that to each pixel of an image is associated an interval which measures the "error" in the value for that pixel. In this situation, therefore, we restrict our set-valued functions so that they only take closed intervals as values. We also need to restrict the ϕ_i maps so that they map intervals to intervals.

Thus, we shall consider $X = [0, 1]^n$ for $n = 1$ or 2 and $Y = [a, b]$. For each x, let $\beta(x) \in \mathcal{H}$ be an interval in Y. Then we define $T : \mathcal{F}(X, Y) \to \mathcal{F}(X, Y)$ by

$$T(F)(x) = \beta(x) + \sum_i p_i(x) \; \alpha_i F(w_i^{-1}(x))$$

where $\alpha_i \in \mathbb{R}$.

Corollary 2. *We have the following inequalities*

$$d_\infty(T(F), T(G)) \leq \left[\sup_x \sum_i \alpha_i p_i(x) \right] d_\infty(F, G)$$

$$d_p(T(F),T(G)) \leq C(n)\left(\sum_i \alpha_i^p s_i^p p_i^p\right)^{1/p} d_p(F,G)$$

where $p_i = \sup_x p_i(w_i(x))$ *and* $s_i \geq 0$ *be such that* $d\mu(w_i(x)) \leq s_i d\mu(x)$.

Proof. We only need to see that

$$d_h\left(\beta(x)+\sum_i p_i(x)\alpha_i F(w_i^{-1}(x)), \beta(x)+\sum_i p_i(x)\alpha_i G(w_i^{-1}(x))\right)$$
$$= d_h\left(\sum_i p_i(x)\alpha_i F(w_i^{-1}(x)), \sum_i p_i(x)\alpha_i G(w_i^{-1}(x))\right)$$

from which point the proof is the same as the proof of Proposition 6.

In Figure 1 are presented the attractor multifunctions for two IFSMF with contractive affine IFS maps w_i. The top image corresponds to the attractor of the following IFSMF

$$\begin{aligned} w_1(x) &= 0.6x, & \phi_1(t) &= 0.7t, \\ w_2(x) &= 0.6x+0.4, & \phi_2(t) &= 0.5t, \\ & 0.5 \leq \beta(x) \leq 1.0. \end{aligned}$$

The right image corresponds to the attractor of the IFSMF with the same w_i and ϕ_i maps but with

$$\begin{aligned} 0 \leq \beta(x) &\leq 1, & 0 &\leq x < 0.5, \\ 0.5 \leq \beta(x) &\leq 1.5, & 0.5 &\leq x \leq 1. \end{aligned}$$

4.1 Fractal block coding and the inverse problem

The inverse problem can be formulated as follows: Given a multifunction $F \in \mathcal{F}(X,Y)$, find a contractive IFSMF operator $T : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$ that admits a unique fixed point $\tilde{F} \in \mathcal{F}(X,Y)$ such that $d_\infty(F,\tilde{F})$ is small enough. As discussed in the introduction, it is in general a very difficult task to find such operators. A tremendous simplification is provided by the "Collage Theorem" [3,1], which we now state with particular reference to IFSMF.

Theorem 1. *(Collage Theorem for IFMSF) Given* $F \in \mathcal{F}(X,Y)$ *suppose that there exists a contractive operator* T *such that* $d_\infty(F,T(F)) < \epsilon$. *If* F^* *is the fixed point of* T *and* $c := \sup_x \sum_i \alpha_i p_i(x)$ *then*

$$d_\infty(F,F^*) \leq \frac{\epsilon}{1-c}$$

The inverse problem then becomes one of finding a contractive IFSMF operator that maps the "target" multifunction F as close to itself as possible.

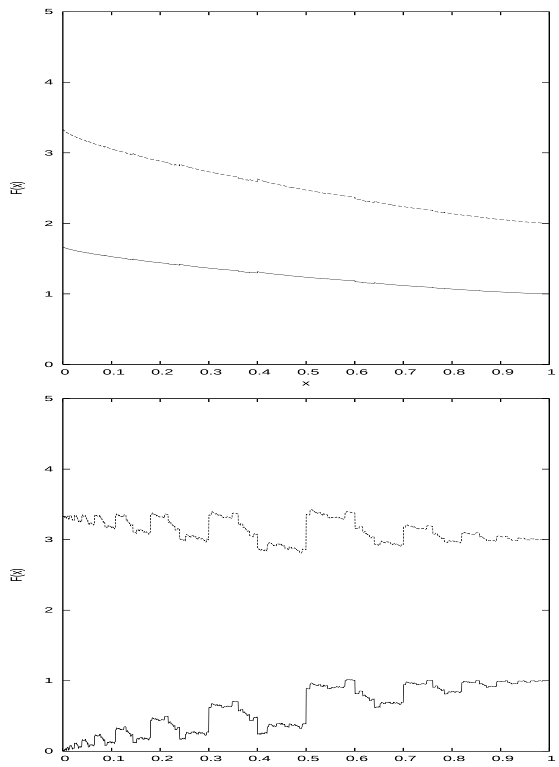

Figure 1. Fixed-point attractor multifunctions $\bar{u}$ for the two IFSMF on $[0, 1]$ given in the text. The upper and lower values of $\bar{u}(x)$ for $x \in [0, 1]$ are sketched.

Corollary 3. *Under the assumptions of the Collage Theorem we have the following inequality*

$$d_\infty(F, TF) \le \sum_i p_i \sup_{x\in X} \max\{\underline{A}_i(x), \bar{A}_i(x)\}$$

where $\underline{A}_i(x) = |\min F(x) - \min(\beta(x) + \alpha_i F(w_i^{-1})(x))|$, $\bar{A}_i(x) = |\max F(x) - \max(\beta(x) + \alpha_i F(w_i^{-1}(x)))|$ *and* $p_i = \sup_{x\in X} p_i(w_i(x))$.

Proof. In fact using a previous result on the Hausdorff distance and recalling that F is a closed interval multifunction,

$$\begin{aligned} d_\infty(F, TF) &= d_\infty(F(x), \beta(x) + \sum_i p_i(x)\alpha_i F(w_i^{-1}(x))) \\ &\le d_\infty(F(x), \sum_i p_i(x)(\beta(x) + \alpha_i F(w_i^{-1}(x)))) \\ &\le \sum_i p_i d_\infty(F(x), \beta(x) + \alpha_i F(w_i^{-1}(x))) \\ &\le \sum_i p_i \sup_{x\in X} \max\{\underline{A}_i(x), \bar{A}_i(x)\} \end{aligned}$$

where $\underline{A}_i(x) = |\min F(x) - \min(\beta(x) + \alpha_i F(w_i^{-1})(x))|$, $\bar{A}_i(x) = |\max F(x) - \max(\beta(x) + \alpha_i F(w_i^{-1}(x)))|$ and $p_i = \sup_{x\in X} p_i(w_i(x))$.

We now prove a similar result for the d_p metric.

Corollary 4. *Under the assumptions of the Collage Theorem we have the following inequality*

$$d_p(F, TF)^p \le \|\min F - \min TF\|_p^p + \|\max F - \max TF\|_p^p$$

Proof. Computing, we have

$$\begin{aligned} d_p(F, TF)^p &= \int_X \left(d_h(F(x), \beta(x) + \sum_i p_i(x)\alpha_i F(w_i^{-1}(x))) \right)^p d\mu(x) \\ &\le \int_X \left| \min F(x) - \min(\beta(x) + \sum_i p_i(x)\alpha_i F(w_i^{-1}(x))) \right|^p d\mu(x) \\ &+ \int_X \left| \max F(x) - \max(\beta(x) + \sum_i p_i(x)\alpha_i F(w_i^{-1}(x))) \right|^p d\mu(x) \\ &= \|\min F - \min TF\|_p^p + \|\max F - \max TF\|_p^p \end{aligned}$$

Most fractal block coding methods are based upon a method originally reported by Jacquin [11]. The pixel array defining the image is partitioned into a set of nonoverlapping *range subblocks* R_i. Associated with with each R_i is a larger *domain subblock* D_i, chosen so that the image function $u(R_i)$ supported on each R_i is well approximated by a greyscale-modified copy of the image function $u(D_i)$. In practice, affine greyscale maps are used:

$$u(R_i) \approx \phi_i(u(w_i(D_i)) = \alpha_i u(w_i(D_i)) + \beta_i, 1 \leq i \leq N$$

where $w_i(x)$ denotes the contraction that maps R_i to D_i (in discrete pixel space, the w_i maps will have to include a decimation that reduces the number of pixels in going from R_i to D_i). The greyscale map coefficients α_i and β_i are usually determined by least squares. The domain blocks D_i are usually chosen from a common *domain pool* $\mathcal{D}$. The domain block yielding the best approximation to $u(R_i)$, i.e., the lowest *collage error*,

$$\Delta_{ij} = \| u(R_i) - \phi_{ij}(u(w_{ij}(D_j)) \|, \quad 1 \leq j \leq M,$$

is chosen for the fractal coding (the L^2 norm is usually chosen).

In Figure 2 is presented the fixed point approximation $\bar{u}$ to the standard 512×512 *Lena* image (8 bits per pixel, or 256 greyscale values) using a partition of 8×8 nonoverlapping pixel blocks ($64^2 = 4096$ in total). The domain pool for each range block was the set of $32^2 = 1024$ 16×16 non-overlapping pixel blocks. (This is not an optimal domain pool – nevertheless it works quite well.) The image $\bar{u}$ was obtained by starting with the seed image $u_0 = 255$ (plain white image) and iterating $u_{n+1} = Tu_n$ to $n = 15$.

Figure 2. The fixed point $\bar{u}$ of the fractal transform operator T described in the main text, designed to approximate the standard 512×512 (8bpp) *Lena* image.

We now consider a simple IFSMF version of image coding, using the partition described above. Since the range blocks R_i are nonoverlapping, all coefficients $p_i(x)$ in our IFSMF operator will have value 1. From the *Lena* image function $u(x)$ used above, we shall construct a multifunction $U(x)$ so that

$$U(x) = [u^-(x), u^+(x)].$$

The approximation of the multifunction range block $U(R_i)$ by $U(D_i)$ then takes the form of two coupled problems

$$\begin{aligned} u^-(R_i) &\approx \alpha_i u^-(w_i(D_i)) + \beta_i^-(R_i), \\ u^+(R_i) &\approx \alpha_i u^+(w_i(D_i)) + \beta_i^+(R_i), \quad 1 \leq i \leq N. \end{aligned}$$

For simplicity, we assume that the $\beta^+(x)$ and $\beta^-(x)$ functions are piecewise constant over each block R_i. For a given domain-range block pair D_i/R_i, we then have a system of three equations in the unknowns α_i, β_i^- and β_i^+. The domain block yielding the best total L^2 collage distance,

$$\begin{aligned} \Delta_{ij} = \quad & \| u^-(R_i) - \alpha_i u^-(w_{ij}(D_j)) - \beta_i^-(R_i) \| \\ & + \| u^+(R_i) - \alpha_i u^+(w_{ij}(D_j)) - \beta_i^+(R_i) \|, \quad 1 \leq j \leq M, \end{aligned}$$

is selected for the fractal code. Corresponding to this fractal code will be the multifunction attractor $\bar{U}(x) = [\bar{u}^-(x), \bar{u}^+(x)]$.

To illustrate, we consider the multifunction constructed from the *Lena* image defined as follows,

$$U_{ij} = [u_{ij} - \delta_{ij}, u_{ij} + \delta_{ij}],$$

where

$$\delta_{ij} = \begin{cases} 0, & 1 \leq i, j \leq 255, \\ 40, & 256 \leq i, j \leq 512, \\ 20, & \text{otherwise.} \end{cases}$$

In other words, the error or uncertainty in the pixel values is zero for the upper left quarter of the image, 20 for the upper right and lower left quarters and 40 for the lower right. In Figure 3 below we show the lower and upper functions, $\bar{u}^-(x)$ and $\bar{u}^+(x)$, respectively, produced by a fractal coding of this multifunction.

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Figure 3. The upper (top) and lower (bottom) functions, $\bar{u}^+$ and $\bar{u}^-$ respectively, of the attractor multifunction $\bar{U}$ produced by the IFSMF fractal coding procedure described in the main text.

References

1. M.F. Barnsley, *Fractals Everywhere*, Academic Press, New York (1989).
2. M.F. Barnsley and S. Demko, Iterated function systems and the global construction of fractals, Proc. Roy. Soc. London Ser. A, **399**, 243-275 (1985).
3. M.F. Barnsley, V. Ervin, D. Hardin and J. Lancaster, Solution of an inverse problem for fractals and other sets, Proc. Nat. Acad. Sci. USA **83**, 1975-1977 (1985).
4. C.A. Cabrelli, B. Forte, U.M. Molter and E.R. Vrscay, Iterated fuzzy set systems: a new approach to the inverse problem for fractals and other sets, J. Math. Anal. Appl. **171**, 79-100, (1992).
5. B. Forte, F. Mendivil and E.R. Vrscay, IFS operators on integral transforms, in *Fractals: Theory and Applications in Engineering*, ed. M. Dekking, J. Levy-Vehel, E. Lutton and C. Tricot, Springer Verlag, London (1999).
6. B. Forte and E.R. Vrscay, Theory of generalized fractal transforms, Fractal Image Encoding and Analysis, NATO ASI Series F, Vol 159, ed. Y.Fisher, Springer Verlag, New York (1998).
7. B. Forte and E.R. Vrscay, Inverse problem methods for generalized fractal transforms, in *Fractal Image Encoding and Analysis*, *ibid*.
8. Hutchinson J.,: Fractals and self-similarity, Indiana Univ. J. Math., 30, 713-747 (1981).
9. S.M. Iacus and D. La Torre, Approximating distribution functions by iterated function systems, Journal of Applied Mathematics and Decision Sciences, **1**, 334-345 (2005).
10. S.M. Iacus and D. La Torre, A comparative simulation study on the IFS distribution function estimator, Nonlinear Analysis: Real World Applications, **6**, 774-785 (2005).
11. A. Jacquin, Image coding based on a fractal theory of iterated contractive image transformations, IEEE Trans. Image Proc. **1**, 18-30 (1992).
12. F. Mendivil and E.R. Vrscay, Correspondence between fractal-wavelet transforms and iterated function systems with grey-level maps, in *Fractals in Engineering: From Theory to Industrial Applications*, ed. J. Levy-Vehel, E. Lutton and C. Tricot, Springer Verlag, London (1997).
13. F. Mendivil and E.R. Vrscay, Fractal vector measures and vector calculus on planar fractal domains, Chaos, Solitons and Fractals **14**, 1239-1254 (2002).
14. R.T. Rockafellar and R.J-B. Wets, *Variational analysis*, Springer Verlag, New York (1998).
15. E.R. Vrscay, A generalized class of fractal-wavelet transforms for image representation and compression, Can. J. Elect. Comp. Eng. **23**, 69-84 (1998).