Contractive multifunctions, fixed point inclusions and iterated multifunction systems

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Abstract. We study the properties of multifunction operators that are contractive in the Covitz-Nadler sense. In this situation, such operators T possess fixed points satisying the relation $x \in Tx$. We introduce an iterative method involving projections that guarantees convergence from any starting point $x_0 \in X$ to a point $x \in X_T$, the set of all fixed points of a multifunction operator T. We also prove a continuity result for fixed point sets X_T as well as a "generalized collage theorem" for contractive multifunctions. These results can then be used to solve inverse problems involving contractive multifunctions. Two applications of contractive multifunctions are introduced: (i) integral inclusions and (ii) iterated multifunction systems.

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1 Introduction

In this paper we are concerned with multifunctions $T : X \rightrightarrows Y$, i.e., set-valued mappings from a space X to the power set 2^Y. In particular, we consider multifunctions that satisfy the following contractivity condition: There exists a $c \in [0, 1)$ such that $d_h(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$, where d_h denotes the Hausdorff metric. From a fundamental theorem of Covitz and Naylor [5], if T is contractive in the above sense, then there exists a fixed point $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$. Note that $\bar{x}$ is not necessarily unique. The set of fixed points of T, to be denoted as X_T, will play an important role in this paper.

We first prove a corollary of the Covitz-Naylor theorem using projections onto sets. This provides a method to construct solutions to the fixed point equation $x \in Tx$, essentially by means of an iterative method that converges to a point $x \in X_T$. We then derive two results that can be viewed as multifunction analogues of those that apply when T is a contractive point-to-point mapping (in which case Banach's fixed point theorem applies), namely: (i) a continuity property of fixed point sets X_T and (ii) "collage theorems" for multifunctions.

These results are important in the inverse problem of approximation by fixed points of contractive mappings [7,8], which we state for the case in which $T : X \to X$ is a point-to-point contraction mapping:

> Given a "target" element $y \in X$, we seek a contraction mapping T with fixed point $\bar{x}$ such that $d(y, \bar{x})$ is as small as possible.

In practical applications, however, it is difficult to construct solutions to this problem. Instead, one relies on the following simple consequence of Banach's fixed point theorem,

$$d(y, \bar{x}) \leq \frac{1}{1-c} d(y, Ty), \tag{1}$$

where c is the contractivity factor of T. In the fractal imaging literature, this result is known as the "collage theorem" [3,1]. Instead of trying to minimize the approximation error $d(y, \bar{x})$, one searches for a contraction mapping T that minimizes the *collage error* $d(y, Ty)$. This has been the basis of most, if not all, fractal image coding methods [6,12]. More recently, it has also been employed in various inverse problems involving differential equations [11]. The results in this paper provide the setup for solving inverse problems involving contractive multifunctions.

We then consider two areas to which the contractive multifunction theory summarized above can be applied. The first is *integral inclusions*. We define a multifunction operator T analogous to the Picard integral operator for first order systems of differential equations. Under appropriate conditions this operator is contractive, guaranteeing the existence of a solution to the integral inclusion.

Secondly, we introduce a method of *iterated multifunction systems* (IMS) over a metric space. This is based on a generalization of a standard point-to-point contraction mapping to a set-valued operator. Hutchinson [9] and Barnsley and Demko [2] showed how systems of contractive maps with associated probabilities – called "iterated function systems" by the latter – acting in a parallel manner either deterministically or probabilistically, can be used to construct fractal sets and measures. Here we define an IMS operator T by the parallel action of a set of contractive multifunctions T_i. Under suitable conditions, T is contractive in the sense defined earlier, implying the existence of a fixed-point multifunction of $\bar{x}$ such that $\bar{x} \in T\bar{x}$.

2 Hausdorff distance: properties and results

In the following we let $d(x, y)$ denote the Euclidean distance. We shall also let $\mathcal{H}(X)$ denote the space of all compact subsets of X and $d_h(A, B)$ the Hausdorff distance between A and B, that is

$$d_h(A, B) = \max\{\max_{x \in A} d'(x, B), \max_{x \in B} d'(x, A)\}, \tag{2}$$

where $d'(x, A)$ is the usual distance between the point x and the set A, i.e.,

$$d'(x, A) = \min_{y \in A} d(x, y). \tag{3}$$

In the following we will denote by $h(A, B) = \max_{x \in A} d'(x, B)$. It is well known that the space $(\mathcal{H}(X), d_h)$ is a complete metric space if X is complete [9].

Lemma 1. *Let (X, d) a metric space.*

1. *For all $x, y \in X$, $C \subset X$ we have*

$$d'(x, C) \leq d(x, y) + d'(y, C). \tag{4}$$

2. *If $A \subset B$ then $h(C, A) \geq h(C, B)$ and $h(A, C) \leq h(B, C)$ for all $C \subset X$.*
3. *For all $x, y \in X$ and $A, B \subset X$ we have*

$$d'(x, A) \leq d(x, y) + d'(y, B) + h(B, A). \tag{5}$$

4. *For all $x \in X$ and $A, B \subset X$ we have*

$$d'(x, A) \leq d'(x, B) + h(B, A). \tag{6}$$

5. *Suppose now that $(X, \|\|)$ be a real normed space and $E \subset X$ be a convex subset of X. Let $A_1, A_2, B_1, B_2 \subset E$ and $\lambda_i \in [0, 1]$ and $\sum_i \lambda_i = 1$. Then*

$$d_h(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) \leq \lambda_1 d_h(A_1, B_1) + \lambda_2 d_h(A_2, B_2). \tag{7}$$

6. *Let $A_i, B_i \subset E$ and $\lambda_i \in [0, 1]$ for $i = 1, 2, \ldots, N$, $\sum_i \lambda_i = 1$. Then*

$$d_h\left(\sum_i \lambda_i A_i, \lambda_i B_i\right) \leq \sum_i \lambda_i \; d_h(A_i, B_i). \tag{8}$$

7. *Let $A, B, C \subset E$, $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$. Suppose that A, B, C are compact and A is convex. Then*

$$d_h(A, \lambda_1 B + \lambda_2 C) \leq \lambda_1 d_h(A, B) + \lambda_2 d_h(A, C). \tag{9}$$

Proof. 1. Computing we have

$$d(x, c) \leq d(x, y) + d(y, c) \tag{10}$$

and then taking the infimum with respect to $c \in C$ we have

$$d'(x, C) \leq d(x, y) + d'(y, C). \tag{11}$$

2. If $A \subset B$ then we have

$$d'(c, B) \geq d'(c, A) \tag{12}$$

and taking the supremum with respect to $c \in C$ we have the thesis. In analogous way one can prove the second inequality.

3. For all $x, y, u, z \in X$ we have
$$d(x,u) \le d(x,y) + d(y,z) + d(z,u) \tag{13}$$
Taking the infimum with respect to $u \in A$ we have
$$d'(x,A) \le d(x,y) + d(y,z) + d'(z,A) \le d(x,y) + d(y,z) + h(B,A) \tag{14}$$
and then the infimum with respect to $z \in B$ we have the thesis.
4. From the previous point, choosing $y \in B$ we have
$$d'(x,A) \le d'(x,y) + h(B,A) \tag{15}$$
and then the thesis follows by taking the infimum with respect to $y \in B$.
5. Computing, we see that
$$\begin{aligned} h(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) &= \max_{a_1,a_2} \min_{b_1,b_2} \|\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 b_1 - \lambda_2 b_2\| \\ &\le \max_{a_1,a_2} \min_{b_1,b_2} [\lambda_1 \|a_1 - b_1\| + \lambda_2 \|a_2 - b_2\|] \\ &= \lambda_1 \max_{a_1} \min_{b_1} \|a_1 - b_1\| + \lambda_2 \max_{a_2} \min_{b_2} \|a_2 - b_2\| \\ &= \lambda_1 h(A_1, B_1) + \lambda_2 h(A_2, B_2). \end{aligned}$$
Similarly we have that $h(\lambda_1 B_1 + \lambda_2 B_2, \lambda_1 A_1 + \lambda_2 A_2) \le \lambda_1 h(B_1, A_1) + \lambda_2 h(B_2, A_2)$. Since $h(A_1, B_1) \le d_h(A_1, B_1)$ and $h(B_1, A_1) \le d_h(A_1, B_1)$, we have the desired result.
6. It is easy to see that if A is convex and $\lambda_i \ge 0$ with $\sum_i \lambda_i = 1$ then $A = \sum_i \lambda_i A$. Using this observation we easily get the following lemma.

The following examples state how to calculate the Hausdorff distance when the sets are intervals.

Example 1. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$. Then
$$d_h(A,B) = \max\{|b_1 - a_1|, |b_2 - a_2|\}.$$

Example 2. This example shows that there are no possibilities to prove a result as lemma 1 for the Hausdorff distance. In fact, consider $C = [-3, 0]$, $A = [1, 3]$ and $B = [2, 5/2]$. Then
$$d_h(C,A) = \max\{1 - (-3), 3 - 0\} = 4$$
and
$$d_h(C,B) = \max\{2 - (-3), 5/2 - 0\} = 5.$$
So even if $B \subset A$ then $d_h(C,A) \le d_h(C,B)$.

3 Properties of contractive functions

For the benefit of the reader, we mention some important mathematical results which provide the basis for IFS fractal transform methods and fractal-based approximation methods.

Theorem 1. *(Banach) Let (X, d) be a complete metric space. Also let $T : X \to X$ be a contraction mapping with contraction factor $c \in [0, 1)$, i.e., for all $x, y \in X$, $d(Tx, Ty) \le cd(x, y)$. Then there exists a unique $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$. Moreover, for any $x \in X$, $d(T^n x, \bar{x}) \to 0$ as $n \to \infty$.*

A simple triangle inequality along with Banach's theorem yields the following result.

Theorem 2. *("Collage Theorem" [3,1]) Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \le \frac{1}{1-c} d(x, Tx), \tag{16}$$

where $\bar{x}$ is the fixed point of T.

Another manipulation of the triangle inequality involving x, Tx and $\bar{x}$ yields the following interesting result.

Theorem 3. *("Anti-collage theorem" [13]) Assume the conditions of the Collage Theorem above. Then for any $x \in Y$,*

$$d(x, \bar{x}) \ge \frac{1}{1+c} d(x, Tx), \tag{17}$$

Theorem 4. *("Continuity of fixed points" [4]) Let (Y, d_Y) be a complete metric space and T_1, T_2 be two contractive mappings with contraction factors c_1 and c_2 and fixed points y_1^* and y_2^*, respectively. Then*

$$d_Y(y_1^*, y_2^*) \le \frac{1}{1-c} d_{Y,\sup}(T_1, T_2) \tag{18}$$

where

$$d_{Y,\sup}(T_1, T_2) = \sup_{x \in X} d(T_1(x), T_2(y)) \tag{19}$$

and $c = \min\{c_1, c_2\}$.

4 Contractive multifunctions and fixed point inclusions

We now extend the previous results to the more general case where when set-valued functions (multifunctions) are considered. We recall that a multifunction $T : X \rightrightarrows Y$ is a function from X to the power set 2^Y. We recall that the graph of T is the following subset of $X \times Y$

$$\operatorname{graph} T = \{(x, y) \in X \times Y : y \in T(x)\}. \tag{20}$$

If $T(x)$ is a closed, compact or convex we say that T is closed, compact or convex valued, respectively. A multifunction T is said to be convex if

$$tT(x) + (1-t)T(y) \subset T(tx + (1-t)y) \tag{21}$$

for all $x, y \in X$ and $t \in [0, 1]$. There are two ways to define the inverse image by a multifunction T of a subset M:

1. $T^{-1}(M) = \{x \in X : T(x) \cap M \neq \emptyset\}$
2. $T^{+1}(M) = \{x \in X : T(x) \subset M\}$

The subset $T^{-1}(M)$ is called the *inverse image* of M by T and T^{+1} is called the *core* of M by T. A function $t : X \to Y$ is a selection or selector of T if $t(x) \in T(x)$, $\forall x \in X$. A *fixed point* of a multifunction T satisfies the relation

$$x \in Tx \tag{22}$$

This relation is also known as a *fixed point inclusion*.

The following result gives a condition for the existence of a fixed point of a multifunction T.

Theorem 5. *(Covitz-Nadler [5,10]) Let (X, d) be a complete metric space and suppose that $T : X \to \mathcal{H}(X)$ be a set valued contraction mapping, i.e.*

$$d_h(Tx, Ty) \leq cd(x, y) \tag{23}$$

for all $x, y \in X$ and $c \in [0, 1)$. Then there exists an $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$.

Note that the fixed point $\bar{x}$ is not necessarily unique.

We now prove a corollary of this theorem which will also provide a method to construct solutions of the fixed point equation (22). Our proof is based on the projection of a point onto a set.

Given a point $x \in X$ and a compact set $A \subset X$ we know that the function $d(x, a)$ has at least one minimum point $\bar{a}$ when $a \in A$. So we have

$$d(x, \bar{a}) \leq d(x, a) \tag{24}$$

for all $a \in A$. We call $\bar{a}$ the *projection* of the point x on the set A and denote it as $\bar{a} = \pi_x A$. Obviously $\bar{a}$ is not unique but we choose one of the minima.

We now define the following projection function associated with a multifunction T:

$$P(x) = \pi_x(Tx). \tag{25}$$

Theorem 6. *Let (X, d) be a complete metric space and $T : X \to \mathcal{H}(X)$ a contraction multifunction such that $d_h(T(x), T(y)) \leq Kd(x, y)$ for all $x, y \in X$ with $K \in [0, 1)$. Then:*

1. *For all $x_0 \in X$ there exists a point $\bar{x} \in X$ such that $x_{n+1} = P(x_n) \to \bar{x}$ when $n \to +\infty$.*

2. $\bar{x}$ is a fixed point, that is, $\bar{x} \in T\bar{x}$.

Proof. Starting from a point $x_0 \in X$, take the projection $P(x_0)$ of the point on the set Tx_0. Computing, we have $d'(x_0, Tx_0) = d(x_0, P(x_0))$. Let $x_1 = P(x_0)$ and take the projection of x_1 on the set Tx_1; we have

$$\begin{aligned} d(x_2, x_1) = d(P(x_1), x_1) = d'(x_1, Tx_1) &= d'(P(x_0), Tx_1) \\ &\leq h(Tx_0, Tx_1) \leq cd(x_0, x_1) = cd'(x_0, Tx_0) \end{aligned}$$

and for each $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) = d'(x_n, Tx_n) \leq c^n d'(x_0, Tx_0) = c^n d(x_0, x_1). \tag{26}$$

Then for all $n, m \in \mathbb{N}$, $n_0 \leq n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} c^{n+i} d(x_0, x_1) \\ &= c^n d(x_0, x_1) \sum_{i=0}^{m-1} c^i = c^n \left(\frac{1 - c^m}{1 - c} \right) d(x_0, x_1) \\ &\leq \frac{c^{n_0}}{1 - c} d(x_0, x_1). \end{aligned}$$

Therefore the sequence $\{x_n\}$ is Cauchy in X. Since X is complete there exists an $\bar{x} \in X$ such that $x_n \to \bar{x}$. Since

$$\begin{aligned} d'(\bar{x}, T\bar{x}) &\leq d(\bar{x}, x_n) + d'(x_n, Tx_n) + h(Tx_n, T\bar{x}) \\ &\leq d(\bar{x}, x_n) + c^n d(x_0, Tx_0) + cd(x_n, \bar{x}) \\ &= (1 + c) d(\bar{x}, x_n) + c^n d(x_0, Tx_0), \end{aligned}$$

it follows that $\bar{x} \in T\bar{x}$.

Remark 1. Suppose that (X, d) is a compact metric space and $T : X \to \mathcal{H}(X)$ a nonexpansive multifunction such that $d_h(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. In this case the previous result can be extended taking into account only the sequence $x_{n+1} = P(x_n)$ for which $d(x_{n+1}, x_n) = d(P(x_n), x_n)$ converge to zero. Under this hypothesis and taking $x_0 \in X$, it is easy to prove there exists a subsequence n_k such that $x_{n_k} \to \bar{x}$ when $k \to +\infty$ and $\bar{x}$ is a fixed point.

The following two results provide possible methods of finding solutions of the fixed point inclusions when some hypotheses are added.

Corollary 1. *Let (X, d) be a complete metric space and $T : X \to \mathcal{H}(X)$ a contraction multifunction. Suppose that there exist two selectors t_1 and t_2 of T which are contractions with factors K_1,K_2 and fixed points x_1,x_2, respectively. If T is convex then $x = tx_1 + (1 - t)x_2$ is a fixed point for each $t \in [0, 1]$.*

Proof. Computing, we have

$$\begin{aligned} x &= tx_1 + (1-t)x_2 \in tT(x_1) + (1-t)T(x_2) \\ &\subseteq T(tx_1 + (1-t)x_2) = T(x). \end{aligned}$$

Corollary 2. *Let $X = [a, b]$ and d be the usual Euclidean distance. Suppose that $T : X \to \mathcal{H}(X)$ is a contraction multifunction and that $T(x)$ is convex for each $x \in X$. Then $\min T(x)$ ($\max T(x)$) is a contraction on X.*

Proof. From the hypotheses, $T(x)$ is a subinterval of $[a, b]$ and so

$$\begin{aligned} |\min T(x) - \min T(y)| &\le d_h(T(x), T(y)) \\ &\le \max\{|\min T(x) - \min T(y)|, \\ &\quad |\max T(x) - \max T(y)|\} \\ &\le K d(x, y). \end{aligned}$$

Given a multifunction $T : X \rightrightarrows X$ let $X_T = \{x \in X : x \in Tx\}$ be the set of all fixed points of T. The following results give some properties of the set X_T.

Theorem 7. *Let (X, d) be a complete metric space. Then X_T is complete.*

Proof. Let $x_n \in X_T$ be a Cauchy sequence of points of X_T. Since $x_n \in X$ and X is complete then there exists a point $\bar{x} \in X$ such that $d(x_n, \bar{x}) \to 0$. Now we have

$$d(\bar{x}, T\bar{x}) \le d(\bar{x}, x_n) + d'(x_n, Tx_n) + h(Tx_n, T\bar{x}) \le 2d(\bar{x}, x_n) \tag{27}$$

and the desired result follows.

Remark 2. Let (X, d) be a compact metric space. In this case it is easy to prove that X_T is compact.

Theorem 8. *(Generalized Collage Theorem) Let (X, d) be a complete metric space and $T : X \to \mathcal{H}(X)$ a contraction multifunction with contractivity factor $c \in [0, 1)$. Then for all $x \in X$ there exists a fixed point $\bar{x}$ such that*

$$d(x_0, \bar{x}) \le \frac{d'(x, Tx)}{1-c} \tag{28}$$

so that

$$d'(x, X_T) \le \frac{d'(x, Tx)}{1-c}. \tag{29}$$

Proof. For any $x \in X$, let $x_0 = x$. From Theorem 6, there is exists a point $\bar{x} \in X_T$ such that the projection scheme $P(x_n) \to \bar{x}$. Then

$$\begin{aligned} d(x_0, \bar{x}) &\le \sum_{i=1}^{n} d(x_i, x_{i-1}) + d(x_n, \bar{x}) \\ &= \sum_{i=1}^{n} d(P(x_{i-1}), x_{i-1}) + d(x_n, \bar{x}) \end{aligned}$$

$$\le \sum_{i=0}^{n-1} c^i d(x_1, x_0) + d(x_n, \bar{x})$$
$$\le \sum_{i=0}^{n-1} c^i d'(x_0, Tx_0) + d(x_n, \bar{x})$$
$$\le \frac{1}{1-c} d'(x_0, Tx_0) + d(x_n, \bar{x}).$$

In the limit $n \to \infty$, we have the desired result.

Theorem 9. *(Generalized Anti-Collage Theorem) Let (X, d) be a complete metric space and $T : X \to \mathcal{H}(X)$ a contraction multifunction with contractivity factor $c \in [0, 1)$. Let $X_T = \{x \in X : x \in Tx\}$ be the set of all fixed points of T. Then*

$$d'(x, Tx) \le (1 + c)d'(x, X_T). \tag{30}$$

Proof. From lemma 1 we have that for all $y \in X_T$

$$\begin{aligned} d'(x, Tx) &\le d(x, y) + d'(y, Tx) \\ &\le d(x, y) + h(Ty, Tx) \le (1 + c)d(x, y). \end{aligned}$$

The desired result follows by taking the infimum.

The following result establishes the continuity or stability of the fixed point set X_T of a contractive multifunction.

Theorem 10. *Let (X, d) be a complete metric space and $T_1, T_2 : X \rightrightarrows X$ be two contractive multifunctions on X. Suppose that X_{T_1} and X_{T_2} are compact sets. Define the following distance*

$$d_\infty(T_1, T_2) = \sup_{x \in X} d_H(T_1x, T_2x). \tag{31}$$

Then

$$d_h(X_{T_1}, X_{T_2}) \le \frac{d_\infty(T_1, T_2)}{1 - \min\{c_1, c_2\}}. \tag{32}$$

Proof. From lemma 1 we know that for all $x \in X$ and $A, B \subset X$ we have

$$d'(x, A) \le d'(x, B) + h(B, A). \tag{33}$$

Let $x \in X_{T_1}$ and computing we have

$$d'(x, T_2x) \le d'(x, T_1x) + h(T_1x, T_2x) \le d_\infty(T_1, T_2) \tag{34}$$

Now using collage theorem we have

$$(1 - c_2)d'(x, X_{T_2}) \le d'(x, T_2x) \le d_\infty(T_1, T_2) \tag{35}$$

and taking the supremum with respect to $x \in X_{T_2}$ we have

$$(1 - c_2)h(X_{T_1}, X_{T_2}) \le d_\infty(T_1, T_2). \tag{36}$$

Upon interchanging X_{T_1} with X_{T_2} we have

$$d_h(X_{T_1}, X_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \min\{c_1, c_2\}}. \tag{37}$$

Corollary 3. *Let $T_n : X \rightrightarrows \mathcal{H}(X)$ be a sequence of contraction multifunctions with contractivity constants such that $\sup_n c_n = S < 1$. Suppose that $T_n \to T$ in the d_∞ metric where $T : X \rightrightarrows \mathcal{H}(X)$ is a contraction multifunction with contractivity factor c. If X_{T_n} and X_T are compacts then $X_{T_n} \to X_T$ in the Hausdorff metric.*

5 Applications: Integral inclusions and iterated multifunction systems

We now apply the results of the previous sections to some examples concerning integral inclusions and iterated multifunction systems (IMS).

5.1 Integral inclusions

Consider now the space of all continuous functions $C([a, b])$ endowed by the classical d_∞ metric. It is well known that $(C([a, b]), d_\infty)$ is a complete metric space. Now consider for each $u \in C([a, b])$ the following operator

$$Tu(x) = \sum_i P_i \int_a^x \phi_i(s, u(s))ds + P_0 := \sum_i P_i \xi_i^u(x) + P_0 \tag{38}$$

where

$$\xi_i^u(x) = \int_a^x \phi_i(s, u(s))ds, \tag{39}$$

P_0 is a compact set, $\phi_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants $K_i > 0$ and $P_i \subset \mathbb{R}$ are compact sets. Let $X = C([a, b])$. It is clear that $T : X \rightrightarrows \mathcal{H}(X)$ where the space $\mathcal{H}(X)$ is the space of all compact subsets of X endowed by the metric

$$d_h(A, B) = \max\{\max_{a\in A} \min_{b\in B} d_\infty(a, b), \max_{b\in B} \min_{a\in A} d_\infty(a, b)\} \tag{40}$$

The space $(\mathcal{H}(X), d_h)$ is complete.

Theorem 11. *$T : (X, d) \to (\mathcal{H}(X), d_h)$.*

Proof. We only need to prove that Tu is a compact subset of X. To do this, take a sequence of elements $l_n \in Tu$; then $l_n = \sum_i p_{i,n}\xi_i^u + p_{0,n}$ and then, eventually by extracting subsequences and using the compactness of P_i we have that $p_{i,n} \to p_i \in P_i$. Let $l = \sum_i p_i\xi_i^u + p_0$. Then

$$d_\infty(l_n, l) \leq \sup_{x\in[a,b]} \sum_i |p_{i,n} - p_i||\xi_i^u(x)|$$

$$\leq \sum_i |p_{i,n} - p_i| \sup_{x\in[a,b]} |\xi_i^u(x)| \to 0$$

when $n \to +\infty$.

The following results shows that Tu is Lipschitz on $C([a,b])$.

Theorem 12. *$d_h(Tu,Tv) \leq c d_\infty(u,v)$ for all $u, v \in X$ where $c = (b-a)\sum_i |p_i| K_i$.*

Proof. First of all we observe that

$$d_h(Tu,Tv) = d_h\left(\sum_i P_i\xi_i^u + P_0, \sum_i P_i\xi_i^v + P_0\right) \leq d_h\left(\sum_i P_i\xi_i^u, \sum_i P_i\xi_i^v\right)$$

Computing, we have

$$\begin{aligned}
h\left(\sum_i P_i\xi_i^u, \sum_i P_i\xi_i^v\right) &= \max_{a\in\sum_i P_i\xi_i^u} \min_{b\in\sum_i P_i\xi_i^v} d_\infty(a,b)\\
&= \max_{p_i\in P_i} \min_{p_i^*\in P_i} d_\infty\left(\sum_i p_i\xi_i^v, \sum_i p_i^*\xi_i^u\right)\\
&= \max_{p_i\in P_i} \min_{p_i^*\in P_i} \sup_{x\in[a,b]} \left|\sum_i p_i\xi_i^v(x) - \sum_i p_i^*\xi_i^u(x)\right|\\
&\leq \max_{p_i\in P_i} \min_{p_i^*\in P_i} \sup_{x\in[a,b]} \left|\sum_i p_i\xi_i^v(x) - \sum_i p_i^*\xi_i^v(x)\right|\\
&\quad + \sup_{x\in[a,b]} \left|\sum_i p_i^*\xi_i^v(x) - \sum_i p_i^*\xi_i^u(x)\right|\\
&\leq \max_{p_i\in P_i} \min_{p_i^*\in P_i} \sum_i |p_i - p_i^*| \sup_{x\in[a,b]} |\xi_i^v(x)|\\
&\quad + \sum_i |p_i^*| \sup_{x\in[a,b]} |\xi_i^v(x) - \xi_i^u(x)|\\
&\leq \max_{p_i\in P_i} \sum_i |p_i| \sup_{x\in[a,b]} |\xi_i^v(x) - \xi_i^u(x)|
\end{aligned}$$

Now recalling that

$$\begin{aligned}
|\xi_i^v(x) - \xi_i^u(x)| &= \left|\int_a^x \phi_i(s,v(s))ds - \int_a^x \phi_i(s,u(s))ds\right|\\
&\leq K_i \int_a^x |v(s) - u(s)|ds\\
&\leq K_i(x-a)d_\infty(v,u)
\end{aligned}$$

we have

$$h\left(\sum_i P_i\xi_i^u, \sum_i P_i\xi_i^v\right) \leq \max_{p_i\in P_i} \sum_i |p_i| \sup_{x\in[a,b]} |\xi_i^v(x) - \xi_i^u(x)|$$

$$\leq \sum_i |p_i| \sup_{x\in[a,b]} K_i(x-a)d_\infty(v,u)$$
$$\leq (b-a)\sum_i |p_i|K_i d_\infty(v,u).$$

Theorem 13. *Suppose that* $c = (b-a)\sum_i |p_i|K_i < 1$. *Then the fixed point inclusion*

$$u \in \sum_i P_i \xi_i^u + P_0 \tag{41}$$

has at least a solution $\bar{u} \in X$. *Furthermore, for all* $u_0 \in C([a,b])$ *there exists a point* $\bar{u} \in X$ *such that* $u_{n+1} = P(u_n) \to \bar{u}$ *when* $n \to +\infty$.

In Figure 1, we illustrate some selections of the integral inclusion

$$x(t) \in (Tx)(t) = \int_0^t \left(P_0 + P_1 x(s) + P_2 x^2(s)\right) ds,$$

where each P_i is the interval $[-1,1]$. We select values $a_i \in P_i$ and plot the solution to the corresponding integral equation. We restrict ourselves to $a_i \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$ for each i, so that there are a total of 125 selections plotted in the Figure.

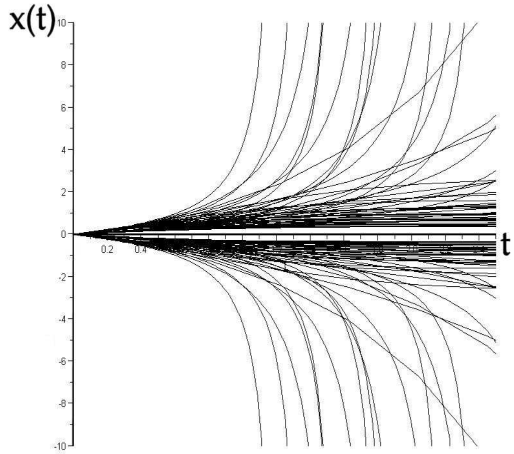

Fig. 1. Some selections of the integral inclusion $x(t) \in (Tx)(t) = \int_0^t \left(P_0 + P_1 x(s) + P_2 x^2(s)\right) ds$, where each P_i is the interval $[-1,1]$.

5.2 Iterated function systems and iterated multifunction systems

First of all we introduce the idea of an iterated function system. Once again, (X, d) denotes a complete metric space, typically $[0,1]^n$. Let $\mathbf{w} = \{w_1, \cdots, w_N\}$ be a set of contraction maps $w_i : X \to X$, to be referred to as an N-map IFS. Let $c_i \in [0,1)$ denote the contraction factors of the w_i and define $c = \max_{1 \le i \le N} c_i \in [0,1)$. As before, we let $\mathcal{H}(X)$ denote the set of nonempty compact subsets of X and h the Hausdorff metric. Associated with the IFS maps w_i is a set-valued mapping $\hat{\mathbf{w}} : \mathcal{H}(X) \to \mathcal{H}(X)$ the action of which is defined to be

$$\hat{\mathbf{w}}(S) = \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{H}(X), \tag{42}$$

where $w_i(S) := \{w_i(x), x \in S\}$ is the image of S under w_i, $i = 1, 2, \cdots, N$. It is a standard result that $\hat{\mathbf{w}}$ is a contraction mapping on $(\mathcal{H}(X), h)$ [9]:

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \le c h(A, B), \quad A, B \in \mathcal{H}(X). \tag{43}$$

Consequently, there exists a unique set $A \in \mathcal{H}(X)$, such that $\hat{\mathbf{w}}(A) = A$, the so-called *attractor* of the IFS $\mathbf{w}$. The equation $A = \hat{\mathbf{w}}(A)$ obviously implies that A is self-tiling, i.e. A is union of (distorted) copies of itself. Moreover, for any $S_0 \in \mathcal{H}(X)$, the sequence of sets $S_n \in \mathcal{H}(X)$ defined by $S_{n+1} = \hat{\mathbf{w}}(S_n)$ converges in Hausdorff metric to A.

Example 3. $X = [0,1]$, $N = 2$: $w_1(x) = \frac{1}{3}x$, $w_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then the attractor A is the ternary Cantor set C on $[0,1]$.

As extension of IFS, consider a set of $T_i : X \rightrightarrows X$ of multifunctions where $i \in 1 \dots n$ and $T_i x \in \mathcal{H}(X)$ for all i. We now construct the multifunction $T : X \rightrightarrows X$ where

$$Tx = \bigcup_i T_i x. \tag{44}$$

Suppose that the multifunctions T_i are contractions with contractivity factor $c_i \in [0,1)$, that is,

$$d_h(T_i x, T_i y) \le c_i d(x, y). \tag{45}$$

From the Covitz-Nadler theorem cited earlier, there exists a point $\bar{x} \in T\bar{x}$. Now given a compact set $A \in \mathcal{H}$ consider the image

$$T(A) = \bigcup_{a \in A} Ta \in \mathcal{H}(X). \tag{46}$$

Since $T : (X, d) \to (\mathcal{H}(X), d_h)$ is a continuous function then $T(A)$ is a compact subset of $\mathcal{H}(X)$. So we can build a multifunction $T^* : \mathcal{H}(X) \rightrightarrows \mathcal{H}(X)$ defined by

$$T^*(A) = T(A) \tag{47}$$

and consider the Hausdorff distance on $\mathcal{H}(X)$, that is given two subset $A, B \subset \mathcal{H}(X)$ we can calculate

$$d_{hh}(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d_h(x,y), \sup_{x\in A}\inf_{y\in B} d_h(x,y)\}. \tag{48}$$

We have the following result.

Theorem 14. *$T^* : \mathcal{H}(X) \rightrightarrows \mathcal{H}(X)$ and*

$$d_{hh}(T^*(A), T^*(B)) \leq c d_h(A,B). \tag{49}$$

Proof.

$$\begin{aligned} d_{hh}(T^*(A), T^*(B)) &= \max\{\sup_{x\in T^*(A)}\inf_{y\in T^*(B)} d_h(x,y), \sup_{x\in T^*(B)}\inf_{y\in T^*(A)} d_h(x,y)\} \\ &= \max\{\sup_{a\in A}\inf_{b\in B} d_h(Ta,Tb), \sup_{b\in B}\inf_{a\in A} d_h(Tb,Ta)\} \\ &\leq c\max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(b,a)\} = c d_h(A,B) \end{aligned}$$

We therefore have the following result.

Theorem 15. *Let (X,d) be a complete metric space and $T_i : X \to \mathcal{H}(X)$ be a finite number of contractions with contractivity factors $c_i \in [0,1)$. Let $c = \max_i c_i$. Then*

1. *For all compact $A \subset X$ there exists a compact subset $\bar{A} \subset X$ such that $A_{n+1} = P(A_n) \to \bar{A}$ when $n \to +\infty$.*
2. *$\bar{A} \subset \bigcup_i T_i(\bar{A})$.*

5.3 Iterated multifunction systems on mappings

Consider now the set X of all functions on $[0,1]$ and the set $E = B([0,1])$ built by taking all the functions $u : [0,1] \to [0,1]$. Obviously E is convex and we can consider on this space the metric d_∞, that is

$$d(u,v) = d_\infty(u,v) = \sup_{x\in[0,1]} |u(x) - v(x)| \tag{50}$$

It is well known that (X,d) is a complete metric space. Consider as usual in IFS theory a set of mappings $w_i : [0,1] \to [0,1]$ for all $i = 1 \dots n$ and let $\phi_i : [0,1] \times [a,b] \to [0,1]$ be a family of Lipschitz functions, that is

$$|\phi_i(\nu,\psi) - \phi_i(\nu,\xi)| \leq K_i|\psi - \xi| \tag{51}$$

for all $\nu \in [0,1]$ and $\psi, \xi \in [a,b]$. Consider now the multifunction $\phi_i^* : B([0,1]) \rightrightarrows B([0,1])$ defined by

$$\phi_i^*(u) = \bigcup_{\xi\in[a,b]} \phi_i(u,\xi) \tag{52}$$

It is trivial to prove that $\phi_i^* : B([0,1]) \to \mathcal{H}(B([0,1]))$ is a contraction. Suppose that ϕ^* satisfies

$$d_h(\phi_i^*(u), \phi_i^*(v)) \leq K_i^* d_\infty(u,v) \tag{53}$$

where $K_i^* \in [0,1)$. Consider the following function $T_{\mathbf{a}}u$

$$T_{\mathbf{a}}u = \sum_i p_i \phi_i(u(w_i^{-1}), a_i), \tag{54}$$

where $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} = (a_1, \ldots, a_n)$, $0 \leq p_i \leq 1$ for all i, $\sum_i p_i = 1$, and we suppose that we put zero corresponding to a term in which $w_i^{-1}(x) = \emptyset$. Clearly $T_{\mathbf{a}}u : [0,1] \to [0,1]$ and we can build the following IMS operator

$$Tu = \bigcup_{\mathbf{a} \in [a,b]^n} T_a u = \sum_i p_i \phi_i(u(w_i^{-1}), [a,b]). \tag{55}$$

Theorem 16. *Tu is compact in $(B([0,1]), d_\infty)$.*

Proof. To prove this, take a sequence of elements $v_k \in Tu$, then $v_k = T_{\alpha_k}u$ but $\alpha_k \in [a,b]^n$ and so, eventually by extracting subsequences, we have $\alpha_k \to \alpha$. We now prove that $v_k \to T_\alpha u$. Computing, we have

$$\begin{aligned}
d_\infty(v_k, T_\alpha u) &= \sup_{x\in[0,1]} |v_k(x) - T_\alpha u(x)| \\
&= \sup_{x\in[0,1]} |T_{\alpha_k}u(x) - T_\alpha u(x)| \\
&\leq \sup_{x\in[0,1]} \sum_i p_i |\phi_i(u(w_i^{-1}(x)), \alpha_i) - \phi_i(u(w_i^{-1}(x)), (\alpha_k)_i)| \\
&\leq \sum_i p_i K_i |\alpha_i - (\alpha_k)_i|.
\end{aligned}$$

So $T : B([0,1]) \rightrightarrows B([0,1])$ and $Tu \in \mathcal{H}(B([0,1]))$. We now prove that T is a contraction.

Theorem 17. *The multifunction $T : B([0,1]) \rightrightarrows B([0,1])$ satisfies*

$$d_h(Tu, Tv) \leq c d_\infty(u,v), \tag{56}$$

where $c = \sum_i p_i K_i^*$.

Proof. Computing we have

$$\begin{aligned}
d_h(Tu, Tv) &= d_h(\sum_i p_i \phi_i(u(w_i^{-1}), [a,b]), \sum_i p_i \phi_i(v(w_i^{-1}), [a,b])) \\
&\leq \sum_i p_i d_h(\phi_i(u(w_i^{-1}), [a,b]), \phi_i(v(w_i^{-1}), [a,b])) \\
&= \sum_i p_i d_h(\phi_i^*(u(w_i^{-1})), \phi_i^*(v(w_i^{-1})))
\end{aligned}$$

$$\leq \sum_i p_i K_i^* d_\infty(u(w_i^{-1}), v(w_i^{-1}))$$
$$\leq \sum_i p_i K_i^* d_\infty(u, v).$$

By the previous results we have then proved the following theorem.

Theorem 18. *Let $w_i : [0,1] \to [0,1]$ for all $i = 1 \dots n$ be a set of mappings on $[0,1]$ and let $\phi_i : [0,1] \times [a,b] \to [0,1]$ be a family of Lipschitz functions, that is*

$$|\phi_i(\xi, \alpha) - \phi_i(\xi, \beta)| \leq K_i |\alpha - \beta|. \tag{57}$$

and

$$d_h(\phi_i^*(u)), \phi_i^*(v)) \leq K_i^* d_\infty(u, v) \tag{58}$$

for all $u, v \in B([0,1])$, where $K_i \geq 0$, $K_i^* \in [0,1)$, $\alpha, \beta \in [a,b]$, $\xi \in [0,1]$ and

$$\phi_i^*(u) = \bigcup_{\xi \in [a,b]} \phi_i(u, \xi). \tag{59}$$

Let

$$T_{\mathbf{a}} u = \sum_i p_i \phi_i(u(w_i^{-1}), a_i). \tag{60}$$

where $0 \leq p_i \leq 1$ for all i, $\sum_i p_i = 1$, and

$$Tu = \bigcup_{\mathbf{a} \in [a,b]^n} T_{\mathbf{a}} u \tag{61}$$

Then

1. *For all functions $u_0 \in B([0,1])$ there exists a function $\bar{u} \in B([0,1])$ such that $u_{n+1} = P(u_n) \to \bar{u}$ when $n \to +\infty$.*
2. *$\bar{u} \in T(\bar{u})$.*

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