IFS-Type Operators on Integral Transforms

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Abstract. Most standard fractal image compression techniques rely on using an IFS operator directly on the image function. Sometimes, however, it is more convenient to work on a faithful representation of the image which, in certain applications, may be a transformed version of the image. For example, if an MRI image is scanned in as frequency data it may be more natural to work on the Fourier transform of the image rather than on the image itself. After a brief introduction to fractal transforms and classical fractal image compression, we discuss some generalities of IFS operators on transform spaces. We then illustrate with examples from Fourier, wavelet and Lebesgue transforms.
We emphasize that the operations can be done completely in the transform domain. In some applications, e.g. measures, we may not even need or desire to return to the spatial domain.

1 Introduction

In this paper we consider the construction of fractal, IFS-type (for Iterated Function Systems) operators on transforms of functions. This represents a continuation of earlier work on *generalized fractal transforms* acting on complete metric spaces, e.g. probability measures [10], moments of probability measures [6], fuzzy sets [2], L^p spaces [5], distributions [7], Fourier transforms [8] and discrete wavelet transforms [12, 16]. The "spirit of IFS" permeates this work as in the past. We also note a related work on Fourier and Laplace transforms of fractal sets and multifractal measures [9].

First, we begin with an appropriate complete metric space, $(\mathcal{F}, d_{\mathcal{F}})$, the elements of which may represent "images" defined over a compact metric space (X, d), the *base* or pixel space. For an element $y \in \mathcal{F}$, construct a set of *fractal components* of y, denoted by y_i, which are distorted copies of y supported on subsets $X_i = w_i(X)$, where $w_i : X \to X$ denote IFS contraction maps. Then define an associated *fractal transform* $T : \mathcal{F} \to \mathcal{F}$ that combines the fractal components y_i, $1 \leq i \leq N$, in a suitable manner so that $Ty \in \mathcal{F}$. Under certain (not too restrictive) conditions, T will be contractive on $(\mathcal{F}, d_{\mathcal{F}})$, implying that there exists a unique $\bar{y} \in \mathcal{F}$ such that $T\bar{y} = \bar{y}$.

Given an element $y \in \mathcal{F}$, the *inverse problem of fractal approximation* involves finding an operator $\mathcal{F}$ whose fixed point $\bar{y}$ approximates y to some prescribed accuracy, i.e. given an $\varepsilon > 0$, find a contractive map T_ε such that $d_{\mathcal{F}}(\bar{y}_\varepsilon, y) < \varepsilon$ where $T_\varepsilon \bar{y}_\varepsilon = \bar{y}_\varepsilon$. In practical applications, e.g. fractal image compression [4, 11], it is the operator T_ε that is stored in computer memory. Given any $y_0 \in \mathcal{F}$ (for example, a blank screen), Banach's Fixed Point Theorem implies that the sequence $y_n = T_\varepsilon y_{n-1}$ converges to $\bar{y}_\varepsilon$, the suitable approximation to y.

In most practical applications, in particular image compression, fractal transforms are applied directly to the image. The space $(\mathcal{F}, d_{\mathcal{F}})$ is an appropriate function space, typically $L^2([0,1]^2, m)$. However, as is done in other approximation schemes or image compression methods, one may work with faithful representations of images in a kind of "dual space" $(\mathcal{G}, d_{\mathcal{G}})$. Examples are moments of probability measures [6], Fourier transforms of L^2 functions [8], and discrete wavelet expansions of L^2 functions [3, 12, 15, 16]. In these cases, the IFS operator T on $(\mathcal{F}, d_{\mathcal{F}})$ induces an affine operator M on $(\mathcal{G}, d_{\mathcal{G}})$ as indicated in the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "T"'] & \mathcal{G} \arrow[d, "M"] \\
\mathcal{F} \arrow[r] & \mathcal{G}
\end{tikzcd}$$

Inverse problems of fractal approximation in $(\mathcal{F}, d_{\mathcal{F}})$ are then transformed to inverse problems in $(\mathcal{G}, d_{\mathcal{G}})$. Under the conditions that T, hence M, is contractive, solutions to the inverse problem in $(\mathcal{G}, d_{\mathcal{G}})$ using the Collage Theorem may then be formulated. Indeed, this was the procedure followed in Refs. [6, 8, 16].

We also mention that once function approximation/image compression is achieved in the space $\mathcal{G}$, it may not be necessary to return to the space $\mathcal{F}$. In fact, in some cases, e.g. moments of measures/images, a return may not be practically possible, nor would it be of interest to do so. In such cases, one works exclusively in the transform domain. We emphasize, however, that when working in such "dual spaces," it is important to establish a number of properties, including

1. Completeness of the dual space $(\mathcal{G}, d_{\mathcal{G}})$.
2. The "faithfulness" of $\mathcal{G}$ (i.e. is it an isomorphism of $\mathcal{F}$?) as well as the operator M (i.e. does M map $\mathcal{G}$ to itself?)

Assuming that T is contractive, the above properties are necessary to ensure the existence of a unique fixed point $\bar{g} \in \mathcal{G}$, i.e. $\bar{g} = M\bar{g}$, by Banach's Fixed Point Theorem.

A natural question that arises in the study of induced operators is, "How does operating on $\mathcal{G}$ instead of on $\mathcal{F}$ relate to operating directly on $\mathcal{F}$?" For example, what is "self-similarity" in the transform domain if, in fact, this is a meaningful question?

Although it appears very natural to consider the induced operator M on the space $\mathcal{G}$, we are not constrained to use it. As we shall see below, it may be advantageous to use another operator, depending upon the application. Nevertheless, the conditions of contractivity as well as completeness listed above must still be established.

In this paper $(\mathcal{G}, d_{\mathcal{G}})$ will be an appropriate space of *function transforms*, in particular *integral transforms*. There are two major motivations for this approach:

1. In many cases, the data which we seek to represent or compress is the result of an integral transform on some function space, e.g. MRI data, blurred images.
2. It may be more convenient to work in certain spaces of integral transforms. For example, as we show below, Lebesgue transforms of normalized nonnegative L^1 functions are nondecreasing and continuous functions. These latter functions may be easier to work with, especially in the sense of approximability.

In Section 2, standard IFS-fractal transforms on functions are very briefly reviewed, mostly for purposes of notation. In Section 3, we consider the integral transform, with kernel K, of a "fractally transformed" function, i.e. Tf, and relate it to the integral transform of f. This equation simplifies if K satsifies a general functional equation. It is then of interest to examine whether the kernel K itself can satisfy an IFS-type equation, for which it is necessary to examine the general space of kernels. A special class of solutions for this functional equation are considered. Finally, we present some examples.

For ease of notation and clarity of discussion, the following discussion is retricted to the one-dimensional case. However, the extension to two (or more) dimensions is straightforward.

2 Iterated Function Systems with Maps (IFSM)

Briefly, standard fractal image compression schemes are variations of affine IFSM with associated fractal transform operators [5, 7]. Let (X, d) denote the base or pixel space and m be a measure on X (usually the unit square with Lebesque measure for the computer screen). Let $w_i : X \to X$, $1 \le i \le N$ be contraction maps and denote $X_i = \{w_i(x) | x \in X\}$. (In most practical applications, $w_i : D_i \to R_i$, where $D_i, R_i \in X$ denote, respectively, *domain* and *range* blocks so that $X_i = R_i$.) For simplicity, as is done in practice, the w_i are assumed to be affine contractions. In our one-dimensional treatment, $X = [0, 1]$ and we denote:

$$w_i(x) = c_i x + a_i, \qquad c_i, a_i \in \mathbf{R}, \quad 0 \le c_i < 1, \qquad 1 \le i \le N. \tag{1}$$

Associated with each IFS map w_i is an affine grey-level map:

$$\phi_i(t) = \alpha_i t + \beta_i, \qquad \alpha_i, \beta_i \in \mathbf{R}. \tag{2}$$

The N-map IFSM, denoted in vector notation as $(\mathbf{w}, \Phi)$, defines a *fractal transform operator* $T : \mathcal{F} \to \mathcal{F}$. The action of T on an image function $u \in L^p(X, m)$ is given by [5, 7]

$$(Tu)(x) = \sum_{i=1}^{N} u_i(x), \tag{3}$$

where the $u_i(x)$ denote the *fractal components* of X:

$$u_i(x) = \begin{cases} \phi_i(u(w_i^{-1}(x))), & x \in X_i, \\ 0, & x \notin X_i. \end{cases} \tag{4}$$

Because of the additivity of integrals, the natural combination of fractal components in L^p spaces is addition. A straightforward calculation yields the bound

$$\| Tu - Tv \|_p \leq C_p \| u - v \|_p, \quad \forall u, v \in L^p(X, m), \tag{5}$$

where

$$C_p = \sum_{i=1}^{N} |c_i|^{1/p} |\alpha_i|. \tag{6}$$

The scalars c_i and α_i are seen to determine whether or not the operator T is contractive in $L^p(X, m)$. If T is contractive, then there exists a unique fixed point $\bar{u} = T\bar{u}$. Note that if all coefficients $\beta_i = 0$, then T is linear in $L^p(X, m)$, implying that $\bar{u} = 0$ is a fixed point.

3 Fractal Transforms of Integral Transforms

3.1 Derivation of a Functional Equation for the Kernel

In this section we let $\mathcal{S} : \mathcal{F} \to \mathcal{G}$ denote an integral transform with kernel $K : X \times \mathbf{R} \to \mathbf{R}$,

$$\widehat{f}(s) = (\mathcal{S}f)(s) = \int_X K(t, s) f(t) \; dt. \tag{7}$$

We shall also write this transform in inner product form as $\mathcal{S}f = < K, f >$.

Let T be an affine IFSM operator as defined in Eq. (3). For an $f \in L^p(X)$, let $g = Tf$. Then the transform $\widehat{g} = \mathcal{S}(g)$ is given by

$$\begin{aligned} \widehat{g}(s) &= \int_X K(t, s) \sum_{i=1}^{N} \left[\alpha_i f(w_i^{-1}(t)) + \beta_i \right] I_{X_i}(t) \; dt \\ &= \sum_{i=1}^{N} \alpha_i \int_{X_i} K(t, s) f(w_i^{-1}(t)) \; dt \; + \; \sum_{i=1}^{N} \beta_i \int_{X_i} K(s, t) dt \\ &= \sum_{i=1}^{N} \alpha_i c_i \int_X K(c_i u + a_i, s) f(u) \; du \; + \; \widehat{\beta}(s), \end{aligned} \tag{8}$$

where

$$\widehat{\beta}(s) = \sum_{i=1}^{N} \beta_i \widehat{I_{X_i}}(s). \tag{9}$$

(Note that $\widehat{\beta}(s)$ depends only on the β_i - and, of course, the X_i - but not on f.)

Eq. (8) may be written in the form

$$< K, Tf > = < T^{\dagger} K, f > \ + \ L(s), \qquad f \in \mathcal{F}, \tag{10}$$

where the operator $T^{\dagger}$ may be interpreted as a kind of "adjoint" fractal operator on the kernel K,

$$(T^{\dagger} K)(t, s) = \sum_{i=1}^{N} \alpha_i c_i K(c_i t + a_i, s), \tag{11}$$

and L as a kind of condensation function. However, the dilations in the spatial variable produced by $T^{\dagger}$ in Eq. (11) represent *expansions.* In contrast to IFSM fractal transforms on functions, the transform K is tiled with expanded copies of itself.

We now focus on Eq. (8) and attempt to rewrite the integrals involving K as *bona fide* integral transforms of f. First, we write

$$\int_X K(c_i u + a_i, s) f(u) du = \int_X \frac{K(c_i u + a_i, s)}{K(u, \zeta_i(c_i, a_i, s))} K(u, \zeta_i(c_i, a_i, s)) f(u) du, \tag{12}$$

where the ζ_i functions perform a renormalization or scaling of the transform variable s. It is desirable that the quotient in the integrand on the right be independent of the integration variable u, i.e. constant with respect to u. Most generally, this implies that

$$K(c_i u + a_i, s) = C_i(c_i, a_i, s) K(u, \zeta_i(c_i, a_i, s)), \qquad i = 1, 2, \ldots, N. \tag{13}$$

However, allowing each scaling relation to possess its own functions C_i and ζ_i may be too general since, for example, no "self-similarity" property is imposed on K . Therefore, we postulate the following functional relation to be satisfied by the kernel K and the functions C and ζ:

$$K(c_i u + a_i, s) = C(c_i, a_i, s) K(u, \zeta(c_i, a_i, s)), \quad \forall u \in X, \qquad i = 1, 2, \ldots, N. \tag{14}$$

Eq. (14) may be considered in several ways, including:

1. as a functional relation between the kernel K, the constant C and scaling function ζ,
2. as a functional equation in the unknown functions K and ζ, given C,
3. as a functional equation in the unknown functions C and ζ, given K.

As in the case of differential equations, the solution of functional equations requires "initial conditions." In addition, however, an admissible space of functions in which solutions are sought must also be specified. This is the subject of future research. A few simple results are presented in Section 3.3.

3.2 Induced Fractal Operators on the Fractal Transforms

If the functional equation in (14) is satisfied by the kernel K, then the integrals in the first sum of Eq. (8) simplify to

$$\sum_{i=1}^{N} \alpha_i c_i \int_X C(c_i, a_i, s) K\left(u, \zeta(c_i, a_i, s)\right) f(u)\, du = \sum_{i=1}^{N} \alpha_i c_i C(c_i, a_i, s) \widehat{f}(\zeta(c_i, a_i, s)).$$

The net result is the relation

$$\begin{aligned} \widehat{g}(s) &= (M\widehat{f})(s) \\ &= \sum_{i}^{N} \alpha_i c_i C(c_i, a_i, s) \widehat{f}(\zeta(c_i, a_i, s)) \;+\; \widehat{\beta}(s), \end{aligned} \tag{15}$$

a kind of self-similarity equation defining the action of operator M in the figure of Section 1.

If we now assume that T is contractive in $L^p(X)$ with fixed point $\bar{f}$, then $\widehat{\bar{f}} = \mathcal{S}(\bar{f})$ satisfies the fixed-point equation $\widehat{\bar{f}} = M\widehat{\bar{f}}$ or

$$\widehat{\bar{f}}(s) = \sum_{i=1}^{N} \alpha_i c_i C(c_i, a_i, s) \widehat{\bar{f}}(\zeta(c_i, a_i, s)) + \widehat{\beta}(s). \tag{16}$$

However, there remains the question whether the operator M is *contractive* in the space of transforms $\mathcal{G}$ and with respect to what metric. Assuming that $\mathcal{G}$ is a Banach space with norm denoted $|| \cdot ||_{\mathcal{G}}$, we define the metric $d_{\mathcal{G}}(u, v) = || u - v ||_{\mathcal{G}}$ for $u, v \in \mathcal{G}$. If M is contractive in this metric then, by Banach's Fixed Point Theorem, $\widehat{\bar{f}}(s)$ may be generated by standard iteration: Start with any function $v_0 \in \mathcal{G}$ and define $v_{n+1} = Mv_n$. Then $v_n \to \widehat{\bar{f}}$ as $n \to \infty$ in the $d_{\mathcal{G}}$ metric.

In many practical examples, including Fourier and wavelet transforms, the coefficients $C(c_i, a_i, s)$ in Eq. (15) may be bounded with respect to s. In such cases, a straightforward calculation yields

$$d_{\mathcal{G}}(Mu, Mv) \le D d_{\mathcal{G}}(u, v), \qquad u, v \in \mathcal{G}, \tag{17}$$

where

$$D = \sum_{i=1}^{N} c_i |\alpha_i \bar{C}_i(c_i, a_i, s)| |J_i|^{-1} \tag{18}$$

where $\bar{C}_i = \max_s C(c_i, a_i, s)$ and $|J_i|$ denotes the (maximum of the) Jacobian of the transformation $s \to \zeta(c_i, a_i, s)$. Contractivity of M is guaranteed if $D < 1$.

3.3 Some Remarks on the Functional Equation for the Kernel

It is natural to inquire about the actual meaning of the functional equation in (14). Suppose that K is a solution. Furthermore, consider the particular case in which the sets $X_i = w_i(X)$ (or the range blocks R_i) $1 \leq i \leq N$, form a partition of X, herewith to be referred to as an *IFS partition* of X, the case normally employed in practical fractal image and signal compression. In this nonoverlapping case, each point $x \in X$ has only one fractal component (neglecting boundary points in the continuous case). As a result, we may "invert" Eq. (14) to give

$$K(t,s) = C(c_i, a_i, s)K(w_i^{-1}(t), \zeta(c_i, a_i, s)), \quad t \in X_i, \qquad i = 1, 2, \ldots, N. \tag{19}$$

The nonoverlapping nature of the X_i allows us to express this result as follows,

$$\begin{aligned} K(t,s) &= (\mathcal{M}K)(t,s) \\ &= \sum_{i=1}^{N} C(c_i, a_i, s)K(w_i^{-1}(t), \zeta(c_i, a_i, s)). \end{aligned} \tag{20}$$

Thus, as in the case of the IFSM fractal transform T, cf. Eq. (3), K is now written as a linear combination of its own *fractal components* under the action of the IFS maps w_i. In other words, K is the fixed point of a fractal transform $\mathcal{M}$ that operates on kernels. Note that there is no restriction on the IFS partition of X, implying that K satisfies a kind of *universal self-similarity.* Clearly, this is a special property.

The following proposition shows that the functional equation in (14) is equivalent to this type of universal self-similarity.

Proposition 1. *The function $K(t,s)$ satisfies the functional equation Eq. (14) for fixed functions $C(c,a,s)$ and $\zeta(c,a,s)$ if and only if for every IFS partition of X with IFS maps of the form $w_i(x) = c_i x + a_i$ there are functions $C_i(s)$ and $\zeta_i(s)$ so that K is the fixed point of the fractal transform operator*

$$(\mathcal{M}K)(t,s) = \sum_{i=1}^{N} C_i(s)K(w_i^{-1}(t), \zeta_i(s)). \tag{21}$$

Proof. By the comments immediately preceding the statement of the proposition, we know that if K satisfies the functional equation, then for any IFSM partition of X, K is the fixed point of the IFSM (21) where $C_i(s) = C(c_i, a_i, s)$ and $\zeta_i(s) = \zeta(c_i, a_i, s)$.

Conversely suppose that for any IFSM partition of X there are functions $C_i(s)$ and $\zeta_i(s)$ such that K is the fixed point of the induced IFSM operator (21). In order to show that K is a solution to the functional equation, we must define the functions $C(c,a,s)$ and $\zeta(c,a,s)$.

To this end, let c and a be fixed such that $w_1(x) = cx + a$ defines a contractive map from X to itself. Choose $w_2, w_3, \ldots, w_n$ to be affine maps such that the IFS $\{w_1, w_2, \ldots, w_n\}$ is an IFS partition of X. Then by hypothesis we know that there are functions $C_i(s)$ and $\zeta_i(s)$ so that K is the fixed point of the induced IFSM given by Eq. (21). Define

$$C(c,a,s) = C_1(s)$$

and

$$\zeta(c, a, s) = \zeta_1(s).$$

Then for all s and t and for this specific choice of c and a we have

$$K(ct + a, s) = C(c, a, s)K(t, \zeta(c, a, s))$$

and so K satisfies the functional equation Eq. (14) for this specific choice of c and a.

Clearly, since c and a were arbitrary, the above procedure can be performed for all c and a, thus constructing functions $C(c, a, s)$ and $\zeta(c, a, s)$ so that K satisfies the functional equation. $\square$

To repeat, the functional equation is equivalent to the property of universal self-similarity. The solution K is the fixed point of an IFSM-type operator on kernel functions. Note, however, that it is *not* guaranteed that the coefficients $C_i(s)$ are contractive. An additional complication arises from the fact that the operator $\mathcal{M}$ in Eq. (20) is linear in K. In order to avoid the trivial solution $K(t, s) = 0$, it may be necessary to restrict the solution space of the functional equation to appropriate "shells," as is done, for example, in the case of IFSP Markov operators and probability measures. These are open questions for further research. We now examine the functional equation for some very special cases.

Proposition 2. *Suppose that a kernel K satisfies the functional equation in Eq. (14) for $\zeta(c_i, a_i, s) = s$. Then K is independent of t, i.e. $K(t, s) = K(s)$.*

Proof. For simplicity of notation, let us drop the subscripts i. Choose fixed values of c and a. Then for $K(t, s) \neq 0$ we have

$$C(c, a, s) = \frac{K(ct + a, s)}{K(t, s)}.$$

Since $t \in X$ and $s \in \mathbf{R}$ are independent variables, it follows that both sides of the equation are independent of t (since the left-hand side is t-independent). Now, choose the value $t^* = a/(1 - c)$ so that $ct^* + a = t^*$. (The existence of such a $t^* \in X$ is guaranteed by the assumption on the IFS maps that $w_i : X \to X$.) Inserting this value of t into the above equation yields

$$C(c, a, s) = \frac{K(t^*, s)}{K(t^*, s)} = 1.$$

This result is true for all values of c, a, s. Therefore, the functional equation reduces to

$$K(ct + a, s) = K(t, s),$$

the only solution of which is $K(t, s) = f(s)$, a function of s only. $\square$

The following result is obtained in a very similar fashion.

Proposition 3. *Suppose that the kernel K satisfying the functional equation in Eq. (14) is independent of s, i.e. $K(t, s) = K(t)$. Then K is a constant.*

These two simple results illustrate the importance of "mixing" between the spatial variable $t \in X$ and the transform variable $s \in \mathbf{R}$. In the following, the particular consequences of separability of the kernel K are examined.

Proposition 4. *Suppose that the kernel K satisfying the functional equation in Eq. (14) is separable, i.e. $K(t,s) = K_1(t)K_2(s)$. Then K_1 is constant on X and K_2 satisfies the relation*

$$K_2(s) = C(c_i, a_i, s)K_2(\zeta(c_i, a_i, s)). \tag{22}$$

Proof. Once again, for simplicity of notation, we omit the subscripts i and choose fixed values of c and a. Then, assuming separability (as well as $K(t,s) \neq 0$), a rearrangement of Eq. (14) yields

$$\frac{K_1(ct+a)}{K_1(t)} = C(c,a,s)\frac{K_2(\zeta(c,a,s))}{K_2(s)} = A, \tag{23}$$

where A is a real constant, since $t \in X$ and $s \in \mathbf{R}$ are independent. For the particular value $t = t^* = a/(1-c)$, we find that $A = 1$, which must hold true for all values of c, a, s. Therefore $K_1(cu+a) = K_1(u)$, implying that K_1 is constant on X. The functional relation (22) for K_2 then follows immediately. □

Proposition 5. *Let $T : L^p(X) \to L^p(X)$ be the fractal transform operator associated with an N-map affine IFSM, as defined in Eq. (3) of Section 2. For an $f \in L^p(X)$, let $g = Tf$. Let $\widehat{f}$ and $\widehat{g}$ denote the integral transforms of f and g respectively, assuming that the kernel K satisfies the functional equation in Eq. (14) and is separable, i.e. $K(t,s) = K_1(t)K_2(s)$. Then*

$$\widehat{g}(s) = \left[\sum_{i=1}^{N} \alpha_i c_i\right] \widehat{f}(s) + \widehat{\beta}(s), \tag{24}$$

where $\widehat{\beta}(s)$ is defined in Eq. (9).

Proof. From the previous proposition, it follows that $K_1(x) = B$, a constant, on X. Therefore,

$$\widehat{f}(s) = BK_2(s)\int_X f(t)dt. \tag{25}$$

Substitution into Eq. (15) yields

$$\widehat{g}(s) = \sum_{i=1}^{N} \alpha_i c_i BK_2(s)\int_X f(t)dt + \widehat{\beta}(s), \tag{26}$$

which, when rearranged, gives the desired result. □

The reader may compare Eq. (24) with Eq. (15). When the kernel K is separable, the resulting operator M relating $\widehat{g}$ to $\widehat{f}$ is rather simple in form, involving no dilations in the transform variable s. (This is a consequence of the fact that K is constant with respect to variations in the spatial variable $t \in X$.) If we further assume that the IFSM operator T is contractive with fixed point $\bar{f}$, then, from Eq. (24), with $f = g = \bar{f}$, it follows that

$$\widehat{\bar{f}}(s) = \widehat{\beta}(s) \left[1 - \sum_{i=1}^{N} \alpha_i c_i \right]^{-1}. \tag{27}$$

4 Examples

4.1 Fourier Transform

The kernel is $K(t, \omega) = e^{\mathbf{i}\omega t}$, so that

$$K(c_i u + a_i, \omega) = e^{\mathbf{i}(c_i u + a_i)\omega} = e^{\mathbf{i}a_i\omega} e^{\mathbf{i}u c_i \omega} = e^{\mathbf{i}a_i\omega} K(u, c_i\omega). \tag{28}$$

Thus, $C(c_i, a_i, \omega) = e^{\mathbf{i}a_i\omega}$ and $\zeta(c_i, a_i, \omega) = c_i\omega$. If $g = Tf$, then Eq. (16) becomes

$$\widehat{g}(\omega) = \sum_i \alpha_i c_i e^{\mathbf{i}a_i\omega} \widehat{f}(c_i\omega) + \widehat{\beta}(\omega). \tag{29}$$

Notice that if T has contractive spatial maps (the w_i's) then the induced operator will have expansive spatial maps. Intuitively, this happens because large frequencies correspond to small scales and small frequencies correspond to large scales. Computationally this happens because the kernel is of the form $K(t, s) = M(st)$. For a kernel of the form $K(t, s) = M(t/s)$, one would have a direct relationship between frequency and scale.

It is well known that $f, g \in L^2(X)$ implies that $\widehat{f}, \widehat{g} \in L^2(\mathbf{R})$. Thus it is convenient to use the usual L^2 metric in $\mathcal{G}$. Following the calculation of Eq. (18), we find

$$\| M\widehat{u} - M\widehat{v} \|_2 \ \le \ \sum_{i=1}^{N} c_i^{1/2} |\alpha_i| \, \| \widehat{u} - \widehat{v} \|_2, \qquad \widehat{u}, \widehat{v} \in L^2(\mathbf{R}). \tag{30}$$

From Eq. (6), with $p = 2$, contractivity of the IFSM operator T implies contractivity of M.

In the case of measures, i.e. $f, g \in \mathcal{M}(X)$, the set of Borel probability measures on X, some care must be taken in the construction of a suitable metric on the space of transforms $\mathcal{G}$ of measures [7]. It can then be shown that contractivity of T implies contractivity of M. We refer the reader to [7] for details.

Finally, from a historical viewpoint, we recall Zygmund's [17] analysis of the Fourier transform of (uniform) Cantor-Lebesgue measure on the classical Cantor set. Not surprisingly, his analysis, which exploited the self-similarity of the problem, was quite analogous to the fractal transform method.

4.2 Wavelet Transform

In this case, the kernel is given by $K(t,s,b) = \psi\left(\frac{t-b}{s}\right)$, where $\psi(x)$ denotes a mother wavelet function. There are two transform variables, s and b, corresponding to scaling and translation, respectively.

$$\begin{aligned} K(c_i u + a_i, s, b) &= \psi\left(\frac{c_i u + a_i - b}{s}\right) \\ &= \psi\left(\frac{u - c_i^{-1}(b - a_i)}{s c_i^{-1}}\right) \end{aligned} \tag{31}$$

so that the functional equation satisfied by K is

$$K(c_i u + a_i, s, b) = K(u, s c_i^{-1}, c_i^{-1}(b - a_i)). \tag{32}$$

Here, $C(c_i, a_i, s, b) = 1$ and the scaling function for the parameter s is $\zeta(c_i, a_i, s, b) = s c_i^{-1}$. Thus, for $g = Tf$, we obtain

$$\widehat{g}(s,b) = \sum_i \alpha_i c_i \widehat{f}\left(s c_i^{-1}, c_i^{-1}(b - a_i)\right). \tag{33}$$

Numerous authors have studied the possibilities of mixing IFS with wavelets (see, for example, [3, 8, 12, 15, 16] and references therein) with good results. The multiresolution structure of the wavelet transform makes it an ideal candidate for fractal analysis.

4.3 Lebesgue Transform

$$K(t,s) = \begin{cases} 1, & \text{if } 0 \le t \le s \\ 0, & \text{if } s \le t \le 1. \end{cases}$$

This kernel satisfies the functional equation with

$$K(c_i u + a_i, t) = K(u, \frac{t - a_i}{c_i}),$$

where we recall that $c_i > 0$. Another (perhaps more useful) way to write the Lebesgue transform of f is as

$$\widehat{f}(s) = \int_0^s f(t)\ dt. \tag{34}$$

If we restrict f to be positive (as is the case for image functions) then it may be viewed as a density function on $X = [0,1]$. Then $\widehat{f}(s)$ will be the cumulative distribution function (CDF) for f. For $f \in L^1$, $\widehat{f}(s)$ is nondecreasing and continuous, with $(\mathcal{S}f)(0) = 0$. If we assume f to be normalized, i.e. $\| f \|_1 = 1$, then $\widehat{f}(1) = 1$.

If we relax the condition that the Lebesgue transform $\widehat{f}(s)$ be continuous in s, then the space $\mathcal{G}$ is given by

$$\mathcal{G} = \{F : [0,1] \to [0,1] : F(0) = 0, F(1) = 1, F \text{ nondecreasing }\},$$

which is the set of CDFs for probability measures on [0,1]. A suitable choice of fractal-type transforms M on this space is as follows [13, 14]. We again assume that the sets $w_i(X)$ overlap only at their endpoints. Then for an $f \in \mathcal{G}$:

$$(Mf)(x) = \alpha_i f(w_i^{-1}(x)) + \beta_i, \qquad x \in X_i, \tag{35}$$

where

$$\beta_1 = 0 \qquad \text{and} \qquad \alpha_i + \beta_i \leq \beta_{i+1} \leq 1 \qquad \text{and} \qquad \alpha_N + \beta_N = 1.$$

(The reader may verify that the above conditions guarantee that M preserves the nondecreasing property. Technically, the above equation is not valid at any points of intersection of the $w_i(X)$. At those points, one would choose either the maximum or minimum of the two "fractal components" in order to preserve right or left continuity, respectively.) The papers [13, 14] contain an extended discussion of this example with some very nice applications to image representation.

This definition of M on $\mathcal{G}$ is slightly more general than the operator induced by the IFSM map $T : \mathcal{F} \to \mathcal{F}$ as it permits the introduction of point masses at the boundaries of the intervals $X_i = w_i(X)$. It essentially represents a "grand unification" of all IFS-type schemes, since $\mathcal{G}$ may now include the Lebesgue transforms of measures, functions and distributions.

4.4 Moments of Measures

This final example is perhaps more of a "nonexample," in two aspects. First, the domain of the transform is the space of Borel probability measures on $[0,1]$. Second, the transform space is a sequence space – the space of moments of measures on $[0,1]$ (as discussed in [6]).

Let $\mathcal{M}(X)$ be the collection of probability measures on $X = [0,1]$ and $\mu \in \mathcal{M}(X)$. We define the moment sequence by

$$g_k(\mu) = \int_X x^k \, d\mu(x) \quad \text{ for } k = 0, 1, 2, \ldots \tag{36}$$

so the "kernel" for this transform is the function $K(x, n) = x^n$. Clearly, this kernel does not satisfy equation (14).

Note that $\mu_0 = 1$ since μ is a probability measure. Furthermore, $g_{k+1} \leq g_k$ since $x^{k+1} \leq x^k$ for $x \in [0,1]$. Thus, $(g_k) \in l^\infty$. We define the space

$$\bar{l}_0^2 = \{\mathbf{c} = (c_0, c_1, \ldots) \in l^\infty | c_0 = 1\}$$

with the weighted inner product

$$\langle \mathbf{c}, \mathbf{d} \rangle = 1 + \sum_{k=1}^{\infty} \frac{c_k d_k}{k^2}.$$

Then $\bar{l}_0^2$ is a complete metric space and the operator on $\bar{l}_0^2$ induced from the Markov operator on $\mathcal{M}(X)$ is a contraction (see [6]). However, this operator is not of IFS type, since the kernel does not satisfy the functional equation (14).

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