

A classical ergodic property for IFS: A simple proof

B. Forte^{1,2}, F. Mendivil²

¹Facoltà di Scienze MM. FF. e NN. a Cà Vignal
Università Degli Studi di Verona
Strada Le Grazie
37134 Verona, Italy
e-mail: forte@biotech.univr.it

²Department of Applied Mathematics
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
e-mail: mendivil@augusta.math.uwaterloo.ca

(February 12, 2000)

Abstract

Let $\{w_i, p_i\}$ be a contractive IFS with probabilities. We provide a simple proof that for almost every address sequence σ and for all x the limit $\lim_n 1/n \sum_{i \leq n} f(w_{\sigma_n} \circ w_{\sigma_{n-1}} \circ \dots \circ w_{\sigma_1}(x))$ exists and is equal to $\int_X f(z) d\mu(z)$ where μ is the invariant measure of the IFS. This is the so called “ergodic property” for the IFS and was proved by Elton in [3]. However, the uniqueness of the invariant measure was not previously exploited. This provides considerable simplification to the proof.

Let X be a compact metric space and $\{w_i\}_{i=1}^L$ a collection of L contraction maps on X . Let $\{p_i\}$ be a collection of L probabilities (i.e. $\sum_i p_i = 1$).

In [4] Hutchinson proves that there exists a unique measure μ invariant under the Markov operator M defined by

$$M(\nu)(B) = \sum_i p_i \nu(w_i^{-1}(B))$$

where ν is a probability measure on X and B is a Borel subset of X . In fact $M^n(\nu) \rightarrow \mu$ for any probability measure ν since the w_i 's are contractive (see [4, 1]).

The operator U defined as

$$U(f)(x) = \sum_i p_i f(w_i(x))$$

(where x is a point in X and f is a continuous function on X) is the adjoint to M and will play an important role in what follows.

Let

$$\Sigma = \prod_{\mathbf{N}} \{1, 2, \dots, L\}$$

be the code space (see [4, 1]) with P the product measure induced by the measure $p(\{i\}) = p_i$ on each factor.

We will need the projections $\pi^n : \Sigma \rightarrow \Sigma^n$ defined by $\pi^n(\sigma) = (\sigma_n, \sigma_{n-1}, \dots, \sigma_1)$. We will denote $\pi^n(\sigma)$ by σ^n .

For $\alpha^n \in \Sigma^n$ we denote by p_{α^n} the product $p_{\alpha_n} p_{\alpha_{n-1}} \cdots p_{\alpha_1}$. Furthermore, we denote by w_{α^n} the composition

$$w_{\alpha_n} \circ w_{\alpha_{n-1}} \circ \cdots \circ w_{\alpha_1}.$$

The following theorem was proved by Elton in [3]. We provide a simplified proof of this result.

Theorem 1 *For any continuous function f on X and any $x \in X$ we have*

$$\lim_{n \rightarrow \infty} 1/n \sum_{i \leq n} f(w_{\sigma_i} \circ w_{\sigma_{i-1}} \circ \cdots \circ w_{\sigma_1}(x)) = \int_X f(z) d\mu(z)$$

for P almost all address sequences $\sigma \in \Sigma$.

Proof: Let f a continuous function on X and x be a fixed element of X .

We wish to show that

$$1/n \sum_{i \leq n} f(w_{\sigma_i}(x))$$

converges. Let ν -lim be a Banach Limit on $l^\infty(\mathbf{N})$ (see [2], p. 82 for a nice discussion of Banach Limits). Recall that a Banach Limit is a “generalized limit” in the sense that it is a bounded linear functional on l^∞ which does not depend on the first terms of the sequence in l^∞ . We will show that any two Banach limits will give the same value for P almost every $\sigma \in \Sigma$ so that the limit exists almost everywhere.

Now $f \mapsto \nu$ -lim $1/n \sum_{i \leq n} f(w_{\sigma_i}(x))$ is a bounded linear functional on $C(X)$. Thus, by the Riesz Representation Theorem it corresponds to a measure μ_ν on X . Since if $f = 1$, we get the limit equals to 1, we know that this measure is a probability measure.

We show that $\mu_\nu = \mu$ for P almost all σ . This will show that for almost all σ the limit exists and is what we wish it to be.

Let $S : \Sigma \rightarrow \Sigma$ denote the shift map on Σ . Since each w_i is contractive and X is compact, we know that

$$\left| 1/n \sum_{i \leq n} f(w_{\sigma_i}(x)) - 1/n \sum_{i \leq n} f(w_{S(\sigma)^i}(x)) \right| \rightarrow 0$$

as $n \rightarrow \infty$. Thus,

$$\nu\text{-lim}(1/n) \sum_{i \leq n} f(w_{\sigma^i}(x)) = \nu\text{-lim}(1/n) \sum_{i \leq n} f(w_{S(\sigma)^i}(x)).$$

Since the shift map on Σ is ergodic, we know that $\nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_{\sigma^i}(x))$ is constant for P almost all σ .

To show that $\mu = \mu_\nu$ it suffices to show that

$$\int_X f(z) d\mu_\nu(z) = \int_X Uf(z) d\mu_\nu(z)$$

which is the same as showing that

$$\nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_{\sigma^i}(x)) = \nu\text{-lim}_n 1/n \sum_{i \leq n} \sum_j p_j f(w_j \circ w_{\sigma^i}(x))$$

Computing we get

$$\begin{aligned} \nu\text{-lim}_n 1/n \sum_{i \leq n} \sum_j p_j f(w_j \circ w_{\sigma^i}(x)) &= \int_{\sigma \in \Sigma} \nu\text{-lim}_n 1/n \sum_{i \leq n} \sum_j p_j f(w_j \circ w_{\sigma^i}(x)) dP(\sigma) \\ &= \sum_j p_j \int_{\sigma \in \Sigma} \nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_j \circ w_{\sigma^i}(x)) dP(\sigma) \end{aligned}$$

Doing the change of variable $\sigma \equiv (\sigma_1, \sigma_2, \dots) \rightarrow (j, \sigma_1, \sigma_2, \dots)$ we get $dP \rightarrow dP/p_j$ so this integral becomes

$$\begin{aligned} \sum_j p_j/p_j \int_{\sigma_1=j} \nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_{\sigma^{i+1}}(x)) dP(\sigma) &= \int_{\sigma \in \Sigma} \nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_{\sigma^i}(x)) dP(\sigma) \\ &= \nu\text{-lim}_n 1/n \sum_{i \leq n} f(w_{\sigma^i}(x)) \end{aligned}$$

for P almost all σ .

Therefore for P almost all σ we know that μ_ν is invariant under M so $\mu_\nu = \mu$. However, since ν was arbitrary this shows that for P almost all σ

$$\lim_n 1/n \sum_{i \leq n} f(w_{\sigma^i}(x)) = \int_X f(z) d\mu(z)$$

for all f and all $x \in X$. ■

Acknowledgments

The authors are grateful to E.R. Vrscay and C. Sempì for their comments and suggestions. The authors would also like to thank the referees for their suggestions, which did much to improve the paper.

References

- [1] Barnsley, Michael, *Fractals Everywhere*, Academic Press, New York, 1988.
- [2] Conway, John, *A Course in Functional Analysis*, **Graduate Texts in Mathematics; 96**, Springer Verlag, New York, 1990.
- [3] Elton, John, An Ergodic Theorem for Iterated Maps, *Journal of Ergodic Theory and Dynamical Systems* **7** (1987), 481-488.
- [4] Hutchinson, J. E., Fractals and Self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713-747.