Lecture 24

Conservative forces in physics (cont'd)

Determining whether or not a force is conservative

We have just examined some examples of conservative forces in $\mathbf{R^2}$ and $\mathbf{R^3}$. We now address the following question:

Suppose that we are given a force $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$.

1. How do we determine whether or not $\mathbf{F}$ is conservative?
2. And if $\mathbf{F}$ is conservative, how do we find the potential $U : \mathbf{R}^n \to \mathbf{R}$ such that

$$\mathbf{F} = -\vec{\nabla} U. \tag{1}$$

We shall answer these questions by examining the dimensions $n = 1, 2, 3$ separately.

Case 1 $n = 1$**:** This case is straightforward. Any one dimensional force $\mathbf{F} = f(x)\mathbf{i}$ that is a function only of position x is a conservative force. Recall that its associated potential function is defined by

$$U(x) = -\int_{x_0}^{x} f(s)\, ds, \tag{2}$$

where x_0 is a chosen reference point for which $U(0) = 0$.

Case 2 $n = 2$**:** We now deal with forces of the form

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}. \tag{3}$$

If $\mathbf{F}$ is conservative, then by (1),

$$\mathbf{F}(x, y) = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j}. \tag{4}$$

Equating components, we have

$$-\frac{\partial U}{\partial x} = F_1(x, y) \quad \text{(a)}, \qquad\qquad -\frac{\partial U}{\partial y} = F_2(x, y) \quad \text{(b)}. \tag{5}$$

Now differentiate both sides of (a) with respect to y and both sides of (b) with respect to x:

$$-\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial F_1}{\partial y} \quad \text{(c)}, \qquad\qquad -\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial F_2}{\partial x} \quad \text{(d)}. \tag{6}$$

Assuming that U is sufficiently "nice," i.e., its second partial derivatives are continuous, then

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}. \tag{7}$$

This implies that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{8}$$

This is a necessary condition on F_1 and F_2 for $\mathbf{F}$ to be conservative.

Example 1: The force $\mathbf{F} = (3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j}$. Here $F_1(x, y) = 3x^2 - 3y^2$ and $F_2(x, y) = -6xy$. We compute:

$$\frac{\partial F_1}{\partial y} = -6y, \qquad \frac{\partial F_2}{\partial x} = -6y. \tag{9}$$

Therefore $\mathbf{F}$ is conservative.

Example 2: A slight modification of the force in Example 1: $\mathbf{F} = (3x^2 - 3y^2)\mathbf{i} + (4x - 6xy)\mathbf{j}$. Here $F_1(x, y) = 3x^2 - 3y^2$ and $F_2(x, y) = 4x - 6xy$. We compute:

$$\frac{\partial F_1}{\partial y} = -6y, \qquad \frac{\partial F_2}{\partial x} = 4x - 6y. \tag{10}$$

Therefore $\mathbf{F}$ is not conservative.

Example 3: The force $\mathbf{F} = (4x^2 - 4y^2)\mathbf{i} + (8xy - \ln y)\mathbf{j}$. Here $F_1(x, y) = 4x^2 - 4y^2$ and $F_2(x, y) = 8xy - \ln y$. We compute:

$$\frac{\partial F_1}{\partial y} = -8y, \qquad \frac{\partial F_2}{\partial x} = 8y. \tag{11}$$

Therefore $\mathbf{F}$ is not conservative. (It is insufficient that the above partial derivatives are equal on the line $y = 0$. In fact, the original vector field $\mathbf{F}(x, y)$ is not even defined at $y = 0$!)

Let us now return to Examples 1 and 2. You saw how the addition of the term $4x$ to the component $F_2(x, y)$ drastically altered the situation: $\mathbf{F}$ was no longer conservative. However, this probably also led you to think: "Geez, if I added $4y$ to F_2 instead of $4x$, the force $\mathbf{F}$ would still be conservative,

since taking the partial of $4y$ with respect to x would give zero." And if you thought a little more, you would see that adding *any* function $g(y)$ to F_1 and *any* function $f(x)$ to F_1 would still preserve its "conservativeness". In other words, adding any force of the form

$$\mathbf{G}(x, y) = f(x)\mathbf{i} + g(y)\mathbf{j} \tag{12}$$

to $\mathbf{F}$ in Example 1 would produce a new force that is still conservative. Indeed, $\mathbf{G}(x, y)$ is conservative! We encountered such a force in the mass-spring problem examined earlier in this lecture: $f(x) = -k_1 x$ and $g(y) = -k_2 y$.

Determining the potential function associated with a conservative force $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$

(Relevant section from textbook by Adams and Essex: 15.2)

Now suppose that we have determined that a force $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ is conservative. How can we find the associated potential function $U(x, y)$ such that

$$\mathbf{F} = -\vec{\nabla} U \ ? \tag{13}$$

We shall use the planar force in Example 1 above to illustrate the procedure. Since that force is conservative, it follows that there exists a $U(x, y)$ such that

$$(3x^2 - 3y^2)\mathbf{i} - 6xy\mathbf{j} = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j}. \tag{14}$$

Equating components, we have the relations

$$\frac{\partial U}{\partial x} = -3x^2 + 3y^2 \qquad (a), \qquad\qquad \frac{\partial U}{\partial y} = 6xy \qquad (b). \tag{15}$$

From (a), we are looking for a function $U(x, y)$ which, when differentiated partially with respect to x, gives $-3x^2 + 3y^2$. We can "work backwards" by antidifferentiating with respect to x, keeping y fixed:

$$U(x, y) = \int (-3x^2 + 3y^2)\, \partial x = -x^3 + 3xy^2 + g(y). \qquad (c) \tag{16}$$

We have used the notation ∂x to emphasize this partial antidifferentiation respect to x. Note also that the "constant of integration" is an unknown function $g(y)$ since any constant or function of y will be eliminated upon partial differentiation with respect to x. Obviously, we must determine $g(y)$. If we differentiate (c) partially with respect to y:

$$\frac{\partial U}{\partial y} = 6xy + g'(y), \qquad (d) \tag{17}$$

Note that the derivative of g with respect to y is a normal derivative since g was assumed to be a function *only* of y and not x. We now compare (d) and (b):

$$g'(y) = 0, \tag{18}$$

which implies that $g(y) = C$ a constant. The final result is

$$U(x, y) = -x^3 + 3xy^2 + C. \tag{19}$$

This is a one-parameter family of potential functions associated with the conservative force $\mathbf{F}$. You should always check your result:

$$-\frac{\partial U}{\partial x} = 3x^2 - 3y^2 = F_1, \qquad -\frac{\partial U}{\partial y} = -6xy = F_2, \tag{20}$$

so our result is correct.

Note that we used both pieces of information (a) and (b) above. We could have started, however, with (b), partially integrated with respect to y, etc..

The above procedure might seem quite similar to something that you have seen in your course on differential equations (e.g., MATH 228). There, you encountered differentials of the form,

$$M(x, y)dx + N(x, y)dy. \tag{21}$$

If $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, then the above is an exact differential, i.e., there exists a function $U(x, y)$ such that

$$dU = M(x, y)dx + N(x, y)dy. \tag{22}$$

You can then find $U(x, y)$ by partial integration, in the same way as we found the potential $U(x, y)$ above.

Case 3 $n = 3$**:** We now deal with forces of the form

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}. \tag{23}$$

If $\mathbf{F}$ is conservative, then there exists a scalar-valued potential function $U(x, y, z)$ such that $\mathbf{F} = -\vec{\nabla} U$, i.e.,

$$\mathbf{F}(x, y, z) = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} - \frac{\partial U}{\partial z}\mathbf{k}. \tag{24}$$

Equating components, we have

$$-\frac{\partial U}{\partial x} = F_1(x, y) \quad \text{(a)}, \qquad -\frac{\partial U}{\partial y} = F_2(x, y) \quad \text{(b)}, \qquad -\frac{\partial U}{\partial z} = F_3(x, y) \quad \text{(c)}. \tag{25}$$

As we did in the case of $n = 2$, we differentiate both sides of (a) with respect to y and both sides of (b) with respect to x:

$$-\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial F_1}{\partial y} \quad \text{(d)}, \qquad -\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial F_2}{\partial x} \quad \text{(e)}. \tag{26}$$

Because of the equality of the two mixed derivatives, it follows that

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}. \tag{27}$$

But this is not enough, we have to look at the other two possible pairings within (a)-(c). Now differentiate both sides of (a) with respect to z and both sides of (c) with respect to x:

$$-\frac{\partial^2 U}{\partial z \partial x} = \frac{\partial F_1}{\partial z} \quad \text{(f)}, \qquad -\frac{\partial^2 U}{\partial x \partial z} = \frac{\partial F_3}{\partial x} \quad \text{(g)}. \tag{28}$$

Because of the equality of the two mixed derivatives, it follows that

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}. \tag{29}$$

Finally, differentiate both sides of (b) with respect to z and both sides of (c) with respect to y:

$$-\frac{\partial^2 U}{\partial z \partial y} = \frac{\partial F_2}{\partial z} \quad \text{(h)}, \qquad -\frac{\partial^2 U}{\partial y \partial z} = \frac{\partial F_3}{\partial y} \quad \text{(i)}. \tag{30}$$

Because of the equality of the two mixed derivatives, it follows that

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{31}$$

To summarize, the relations that must be satisfied by the three components of a conservative force $\mathbf{F}$ in $\mathbf{R}^3$ are:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{32}$$

These relations, which must hold *simultaneously*, look quite complicated. Later, we'll see that there is a rather simple formula that compactly contains these results.

Finally, how would one determine the potential $U(x, y, z)$ associated with a conservative force $\mathbf{F}$ in $\mathbf{R}^3$? The answer: By tedious, systematic partial integration along the lines of what was done for $\mathbf{R}^2$ earlier. You may find an example of this procedure in the textbook by Stewart: Example 5 in Section 16.3, p. 1051-1052. In this problem, $\mathbf{F}$ is given, and a scalar-valued function f is determined so that $\mathbf{F} = \vec{\nabla} f$.

Very fortunately, we have already encountered the most important conservative forces in $\mathbf{R}^3$: forces of the form

$$\mathbf{F}(\mathbf{r}) = -\frac{K}{r^3}\mathbf{r} \tag{33}$$

for which the associated potential function is

$$U(\mathbf{r}) = -\frac{K}{r}. \tag{34}$$

The two important cases examined earlier are:

1. $K = GMm$: gravitational force exerted by mass M at (0,0,0) on mass m at $\mathbf{r}$,
2. $K = -\dfrac{Qq}{4\pi\epsilon_0}$: electrostatic force exerted by charge Q at (0,0,0) on charge q at $\mathbf{r}$.

Lecture 25

Divergence and Curl of a vector field

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.5)

It is convenient to define the following operator, called the "del" operator,

$$\vec{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{35}$$

This operator could also be written as an ordered triple of operators, i.e.

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{36}$$

Now, an operator acts on suitable mathematical objects. We have already seen an example of the action of the del operator – it can act on scalar-valued functions to produce the *gradient vector*:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{37}$$

You will also see this operator, up to a constant, in your quantum mechanics course. The operator $-\frac{\hbar}{i}\vec{\nabla}$ corresponds to the quantum mechanical *momentum operator*.

Divergence of a vector field

But vectors can also act on vectors. For example, we could form the scalar product of the del operator with a vector field $\mathbf{F}$:

$$\begin{aligned} \vec{\nabla} \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \end{aligned} \tag{38}$$

This is known as the *divergence* of the vector field $\mathbf{F}$. In some books, it is also written as "div $\mathbf{F}$". It is a *scalar quantity*, as should be the case when the dot product of two vectors is taken. Note that we can also define the divergence of vector fields in $\mathbf{R}^2$ by omitting the $\mathbf{k}$ terms. We shall outline a physical interpretation of the divergence in the next lecture.

Examples:

1. $\vec{\nabla} \cdot \mathbf{C} = 0$ for any constant vector $\mathbf{C}(x, y, z) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$.

2. If $\mathbf{F} = x^2yz\ \mathbf{i} + xz^5 \sin y\ \mathbf{j} + x^2y \sin 4z\ \mathbf{k}$, then

$$\vec{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xz^5 \sin y) + \frac{\partial}{\partial z}(x^2y \sin 4z) = 2xyz + xz^5 \cos y + 4x^2y \cos 4z. \tag{39}$$

3. If $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $F_1 = x$, $F_2 = y$ and $F_3 = z$ so that

$$\vec{\nabla} \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3. \tag{40}$$

4. If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$, then $F_1 = -y$, $F_2 = x$ and $F_3 = 0$ so that

$$\vec{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}(0) = 0. \tag{41}$$

Recall that this is the vector field associated with counterclockwise rotation about the z-axis. This vector field also has zero divergence.

5. Now recall the importance of the class of vector fields $\frac{K}{r^3}\mathbf{r}$ in physics. When $K = -GMm$, we have the gravitational force exerted on a mass m at $\mathbf{r}$ by a point mass M at the origin of a coordinate system. Whe $K = Qq/(4\pi\epsilon_0)$, we have the electrostatic force exerted on a charge q at $\mathbf{r}$ due to a point mass Q at the origin. For convenience, we shall omit the multiplicative factor K. We now wish to show that

$$\operatorname{div}\left(\frac{1}{r^3}\mathbf{r}\right) = \vec{\nabla} \cdot \left(\frac{1}{r^3}\mathbf{r}\right) = 0, \qquad (x, y, z) \neq (0, 0, 0). \tag{42}$$

Let us first express this field in terms of Cartesian coordinates:

$$\frac{1}{r^3}\mathbf{r} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}], \tag{43}$$

so that

$$\vec{\nabla} \cdot \frac{1}{r^3}\mathbf{r} = \frac{\partial}{\partial x}\frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2 + y^2 + z^2)^{3/2}}. \tag{44}$$

We evaluate the first partial derivative using the product rule:

$$\begin{aligned} \frac{\partial}{\partial x}\frac{x}{(x^2 + y^2 + z^2)^{3/2}} &= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \left(-\frac{3}{2}\right)\frac{x}{(x^2 + y^2 + z^2)^{5/2}}(2x) \\ &= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}}. \qquad (x, y, z) \neq (0, 0, 0), \end{aligned} \tag{45}$$

Similarly,

$$\frac{\partial}{\partial y}\frac{y}{(x^2 + y^2 + z^2)^{3/2}} = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}}, \qquad (x, y, z) \neq (0, 0, 0), \tag{46}$$

$$\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}, \qquad (x,y,z) \neq (0,0,0), \tag{47}$$

Adding together the above three results yields

$$\frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} = 0, \tag{48}$$

for $(x,y,z) \neq (0,0,0)$. Thus we have the final result,

$$\vec{\nabla} \cdot \frac{1}{r^3}\mathbf{r} = 0, \qquad (x,y,z) \neq (0,0,0). \tag{49}$$

This is an extremely important result! As we'll see later, it reflects how the electrostatic and gravitational fields in $\mathbf{R}^3$ naturally behave in the presence and absence of charges/masses, respectively.

Definition: A vector field $\mathbf{F}$ for which $\vec{\nabla} \cdot \mathbf{F} = 0$ for all points $(x,y,z) \in D \subseteq \mathbf{R}$ is said to be **incompressible** over D. (Some books also use the term **solenoidal**.)

The vector field $\mathbf{F} = \frac{K}{r^3}\mathbf{r}$ examined above is incompressible over the set $\mathbf{R} - \{(0,0,0)\}$.

Curl of a vector field

Let us now take the vector product of the del operator with a vector field $\mathbf{F}$:

$$\begin{aligned}
\vec{\nabla} \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{i} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.
\end{aligned} \tag{50}$$

This is known as the *curl* of the vector field $\mathbf{F}$. In some books, it is also written as " curl $\mathbf{F}$". It is a *vector quantity*, as should be the case when the vector product of two vectors is taken. Note that we can also define the curl of vector fields in $\mathbf{R}^2$ by setting $F_3 = 0$.

Examples:

1. $\vec{\nabla} \times \mathbf{C} = 0$ for any constant vector $\mathbf{C}(x,y,z) = C_1\mathbf{i} + C_2bfj + C_3\mathbf{k}$.

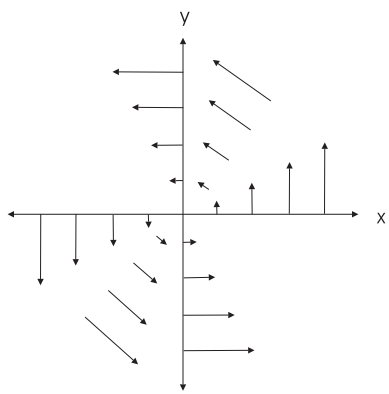
2. If $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $F_1 = x$, $F_2 = y$ and $F_3 = z$ so that

$$\begin{aligned}
\vec{\nabla} \times \mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = \vec{\mathbf{0}}.
\end{aligned}$$

3. Now consider $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$. Then

$$\begin{aligned}
\vec{\nabla} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\
&= \left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(-y)}{\partial z} - \frac{\partial 0}{\partial x}\right)\mathbf{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\mathbf{k} = 2\mathbf{k}.
\end{aligned}$$

We have already encountered this vector field. A view from the positive z-axis looking down onto the xy-plane is shown below.



A top view of the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$

In the xy-plane, it can represent the velocity field of a rotating disk – in this case, angular frequency $\omega = 1$. The above vector field could represent the velocity field of a rotating cylinder. Note that curl $\mathbf{F} = 2\mathbf{k}$ at **all** points in the plane, not just at the origin, where the axis of rotation is situated. This implies that the vector field is "rotational" at all points in the plane. We'll return to this idea later in the course.

4. Once again recall the importance of the class of vector fields $\dfrac{K}{r^3}\mathbf{r}$ in physics. When $K = -GMm$, we have the gravitational force exerted on a mass m at $\mathbf{r}$ by a point mass M at the origin of

a coordinate system. Whe $K = Qq/(4\pi\epsilon_0)$, we have the electrostatic force exerted on a charge q at $\mathbf{r}$ due to a point mass Q at the origin. For convenience, we shall omit the multiplicative factor K. Let us first express this field in terms of Cartesian coordinates:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{r^3}\mathbf{r} = \frac{1}{(x^2+y^2+z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}], \tag{51}$$

$$\vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix} \tag{52}$$

We'll leave this as an exercise for the reader! (It's actually not that tedious.) The net result is

$$\vec{\nabla} \times \frac{1}{r^3}\mathbf{r} = \mathbf{0}, \qquad (x, y, z) \neq (0, 0, 0). \tag{53}$$

Conservative forces and the curl

Let's now return to the definition of the curl of a vector field $\mathbf{F}$

$$\begin{aligned} \vec{\nabla} \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{i} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \end{aligned} \tag{54}$$

Let's also review the conditions that had to be satisfied for a vector field $\mathbf{F}$ to be *conservative*, i.e., that $\mathbf{F} = -\vec{\nabla}U$ for some $U(x, y, z)$:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{55}$$

If these three relations are satisfied, then the three entries of the vector $\vec{\nabla} \times \mathbf{F}$ are zero. In other words,

$$\vec{\nabla} \times \mathbf{F} = \mathbf{0} \quad \text{implies that } \mathbf{F} \text{ is conservative.}$$

In fact, the above relation goes both ways so that "implies that" can be replaced by "if and only if" but the above will be sufficient for our needs. To check whether or not a vector field is conservative, you should examine its *curl*.

The above result may also be expressed in the following way. For any scalar field $f : \mathbf{R}^3 \to \mathbf{R}$,

$$\vec{\nabla} \times (\vec{\nabla} f) = \mathbf{0}. \tag{56}$$

In other words, the curl of the gradient field $\mathbf{F} = \vec{\nabla} f$ is zero: *Gradient or conservative fields are irrotational.* You will be asked to prove this result in an assignment.

Here is another important result that is left as an exercise:

$$\text{div (curl } \mathbf{F}) = \vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{F}) = \mathbf{0}. \tag{57}$$

Lecture 26

Physical interpretation of the divergence

A better understanding of the divergence will be achieved after we study line integrals in the plane. For the moment, let us state that the divergence of a vector field at a point (x, y, z), i.e., $\vec{\nabla} \cdot \mathbf{F}(x, y, z)$ measures the *net outward flow per unit volume* of the field at a point. In what follows, we'll try to give a simple picture to illustrate this fact.

Let us suppose that a fluid (i.e., gas or liquid) is travelling through $\mathbf{R}^3$ and that its velocity at a point $\mathbf{r} = (x, y, z)$ is given by $\mathbf{v}(\mathbf{r})$. The motion of the fluid is therefore determined by the velocity field $\mathbf{v} = (v_1, v_2, v_3)$ over a region $D \subset \mathbf{R}^3$. For simplicity, we shall assume that the field $\mathbf{v}(\mathbf{r})$ does not change over time. Now let P denote a reference with coordinates (x, y, z) and consider a rectangular box B centered at P with sides of infinitesimally small lengths Δx, Δy and Δz, so that its volume is $\Delta V = \Delta x \Delta y \Delta z$. We now ask the question: "What is the net rate of fluid flowing out of the box B, i.e., mass per unit time?" This means that we must also consider the density $\rho(\mathbf{r})$ of the fluid at each point in D. In the case of a liquid, the density, to a very good approximation, is constant. But in the case of a gas, this is not necessarily true.

As such, we must define the **transport** or **flux vector**,

$$\mathbf{u}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}). \tag{58}$$

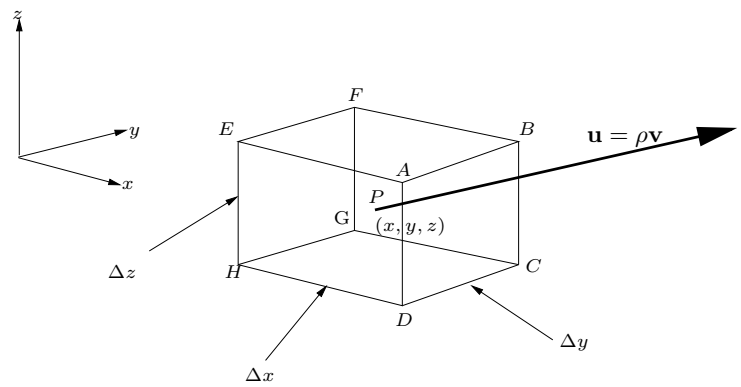
Note that the dimensionality of this vector is

$$(\text{Density}) \times (\text{Velocity}) = \frac{M}{L^3} \cdot \frac{M}{T} = \frac{M}{L^2 T}. \tag{59}$$

If we multiply this vector by a small element of area ΔA (with dimension L^2) perpendicular to it, then we obtain the rate of mass flow through ΔA (dimension $\frac{M}{T}$) through this area. But more on this later.

The box, along with the vector $\mathbf{u}$ emanating from point P, is sketched below. The net rate of fluid outflow is the sum of the following contributions:

1. The net rate of outflow in the x-direction, i.e., the rate of outward flow through side $ABCD$ minus the rate of outward flow through side $EFGH$.



2. The net rate of outflow in the y-direction, i.e., the rate of outward flow through side $BCGF$ minus the rate of outward flow through side $ADHE$.

3. The net rate of outflow in the z-direction, i.e., the rate of outward flow through side $ABFE$ minus the rate of outward flow through side $DCGH$.

We begin with the first contribution – the net outflow in the x-direction. The fact that the vector $\mathbf{u}$ points toward the positive x-direction does not really sacrifice generality in our discussion. It only means that that the outward flow through side $ABCD$ is positive and that the outward flow through $EFGH$ is negative. If $\mathbf{u}$ pointed in the opposite direction, the outward flow through $ABCD$ would be negative and that through $EFGH$ would be positive. There are no problems as long as we maintain consistency.

In the computation of the flow through sides $ABCD$ and $EFGH$, we acknowledge the change in $\mathbf{u}$ that can take place due to a change of Δx in the x-coordinate. We shall, however, ignore any variations in the vector $\mathbf{u}$ over each side, i.e., we assume that there is no change in the y and z directions.

The net amount of fluid flow through side $ABCD$ is determined by the projection of $\mathbf{u}$ in the x-direction, namely v_1: This is the component of $\mathbf{u}$ normal to the side. If v_1 were zero, then no fluid would be flowing across the side. But we must multiply this component by the area of the side to give the total rate of fluid outflow. The net rate of outflow in the x-direction is then given by

$$[v_1(x + \Delta x/2, y, z) - v_1(x - \Delta x/2, y, z)]\Delta y \Delta z. \tag{60}$$

For reasons that will become clearer below, we shall multiply and divide this term by Δx to give

$$\left[\frac{v_1(x + \Delta x/2, y, z) - v_1(x - \Delta x/2, y, z)}{\Delta x}\right] \Delta x \Delta y \Delta z = \left[\frac{v_1(x + \Delta x/2, y, z) - v_1(x - \Delta x/2, y, z)}{\Delta x}\right] \Delta V. \tag{61}$$

In a similar fashion, the net outflow from the box in the y-direction is

$$\left[\frac{v_2(x, y + \Delta y/2, z) - v_2(x, y - \Delta y/2, z)}{\Delta y}\right] \Delta V. \tag{62}$$

And in the z-direction:

$$\left[\frac{v_3(x, y, z + \Delta z/2) - v_3(x, y, z - \Delta z/2)}{\Delta z}\right] \Delta V. \tag{63}$$

We now consider the limits of these expressions as $\Delta x, \Delta y$ and $\Delta z \to 0$. The term in square brackets in Eq. (61) is a Newton quotient representing the change in v_1 due to a change in x of Δx. (It's a little different than the change you considered in first-year calculus – this is a *symmetric difference* about x, between $x + \Delta x/2$ and $x - \Delta x/2$, as opposed to the *forward difference* between $x + \Delta x$ and x of first-year calculus.) In the limit, therefore, it will become the partial derivative $\dfrac{\partial v_1}{\partial x}$. Similarily, the Newton quotients in Eqs. (62) and (63) will become, respectively, the partial derivatives $\dfrac{\partial v_2}{\partial y}$ and $\dfrac{\partial v_3}{\partial z}$. Adding up the three contributions yields the following result for the net outflow through a an infinitesimal rectangular box with volume $dx\ dy\ dz = dV$:

$$\left[\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right] dV = \vec{\nabla} \cdot \mathbf{u}\ dV. \tag{64}$$

This implies that the divergence is the net outward flow per unit volume.

In your course on electricity and magnetism, you will eventually encounter **Maxwell's first equation of electrostatics**,

$$\vec{\nabla} \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}, \tag{65}$$

where $\mathbf{E}(\mathbf{r})$ is the electric field at a point $\mathbf{r}$ and $\rho(\mathbf{r})$ is the charge density at $\mathbf{r}$. In other words, if there is no charge at a point, then the divergence of the field will be zero there. *Charge is responsible for net inflow/outflow (depending upon the sign convention) of electric field.* We shall derive this equation later in this course.

Recall that the electric field at a point $\mathbf{r}$ due to the presence of a point charge Q at the origin is $\mathbf{E} = \dfrac{Q}{4\pi\epsilon_0 r^3}\mathbf{r}$. We showed above that $\vec{\nabla} \cdot \mathbf{E} = 0$ at all points $\mathbf{r} \neq (0, 0, 0)$. At $(0, 0, 0)$, however, the divergence "blows up," i.e., is infinite. This is because the charge density ρ at $(0, 0, 0)$ is infinite: We are assuming that a charge Q exists at a point with no volume. Of course, a point charge is a mathematical abstraction that does not correspond to physical reality. We'll come back to this point later.

Some properties of div and curl

Here are a few properties that follow naturally from the definitions of div and curl which involve derivatives. In the following, $f : \mathbf{R}^3 \to \mathbf{R}$ is a scalar field and $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ is a vector field.

$$\begin{aligned}
\vec{\nabla}(f+g) &= \vec{\nabla}f + \vec{\nabla}g \\
\vec{\nabla}\cdot(\mathbf{F}+\mathbf{G}) &= \vec{\nabla}\cdot\mathbf{F} + \vec{\nabla}\cdot\mathbf{G} \\
\vec{\nabla}\times(\mathbf{F}+\mathbf{G}) &= \vec{\nabla}\times\mathbf{F} + \vec{\nabla}\cdot\mathbf{G} \\
\vec{\nabla}\cdot(f\mathbf{F}) &= f\vec{\nabla}\cdot\mathbf{F} + \mathbf{F}\cdot\vec{\nabla}f \\
\vec{\nabla}\times(f\mathbf{F}) &= f\vec{\nabla}\times\mathbf{F} + \vec{\nabla}f\times\mathbf{F}
\end{aligned}$$

These results can be derived from straightforward application of the definitions. (Some of these are posed as problems in the Exercises of Section 16.5 in the text, page 1068.)

The Laplacian operator

Let f be a scalar field. Taking the gradient of f produces a vector field $\mathbf{F} = \vec{\nabla}f$. We showed above that the curl of $\mathbf{F}$ is the zero vector. But we can also take the divergence of $\mathbf{F}$:

$$\text{div (grad } f) = \vec{\nabla}\cdot(\vec{\nabla}f) = (\vec{\nabla}\cdot\vec{\nabla})f := \nabla^2 f \tag{66}$$

The operator "∇^2" is called the *Laplacian* operator – note once again that it operates on scalar-valued functions. In Cartesian coordinates,

$$\begin{aligned}
\nabla^2 f &= \vec{\nabla}\cdot\vec{\nabla}f \\
&= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

The Laplacian operator is very important in theoretical physics and applied mathematics. You will see it in your courses in electricity and magnetism as well as in quantum mechanics. Just to give you a flavour of its use, let us return to the Maxwell equation for electrostatics given earlier,

$$\vec{\nabla}\cdot\mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{67}$$

In applications, one is often interesting in computing $\mathbf{E}$ from a knowledge of ρ. Recalling that $\mathbf{E}$ is conservative, i.e., $\mathbf{E} = -\vec{\nabla}U$, where U is the associated potential function, we have

$$\vec{\nabla} \cdot (-\vec{\nabla}U) = -\nabla^2 U = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{68}$$

or

$$\nabla^2 U = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{69}$$

This is known as *Poisson's equation*, a second order partial differential equation in $U(x, y, z)$. There exists a huge arsenal of mathematical tools to solve such equations (and you will learn some of these tools in your later courses). In the absence of charge, i.e. $\rho = 0$, Poisson's equation becomes *Laplace's equation*:

$$\nabla^2 U = 0. \tag{70}$$

Of course, you could say that an obvious solution to this equation exists, i.e., $U = 0$. But so are $U = x$, and $U = y$, and an infinity of solutions. The Laplace and Poisson equations are never solved by themselves – there are always *boundary conditions* to be satisfied, e.g. the potential of the outer surface of a sphere enclosing the region of interest. The boundary conditions will isolate particular solutions that are relevant to the problem being studied. This idea is discussed in the Appendix to this lecture for your information: The determination of the potential between two concentric cylinders of different potentials, a problem which you may already have studied in your Physics lab.

You will also encounter the Laplacian operator in quantum mechanics. Recall that the momentum operator was given by $\mathbf{p} = -\frac{\hbar}{i}\vec{\nabla}$. The "square" of this operator, i.e. if you apply this operator twice, is $\mathbf{p} \cdot \mathbf{p} = -\hbar^2 \nabla^2$. Recall that in classical mechanics, the kinetic energy of a particle may be expressed in terms of the square of its momentum as $T = \frac{p^2}{2m}$. Dividing the quantum mechanical operator $\mathbf{p} \cdot \mathbf{p}$ by $2m$ yields the quantum mechanical kinetic energy operator

$$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2, \tag{71}$$

which is a component of the Schrödinger equation for the wavefunction ψ.

The Laplacian operator in the various coordinate systems

We simply state the results below. You will no doubt use these results in future courses in Physics.

1. In Cartesian coordinates, for a function $f(x, y, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{72}$$

2. In cylindrical coordinates, for a function $f(r, \theta, z)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{73}$$

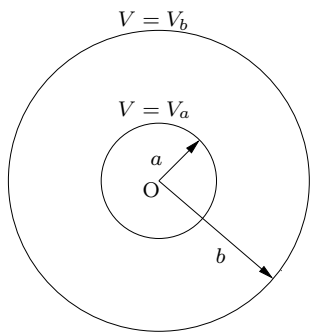
3. In spherical coordinates, for a function $f(r, \theta, \phi)$:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2}\frac{\cos\phi}{\sin\phi}\frac{\partial f}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} \tag{74}$$

Appendix: Solving Laplace's equation for the potential between two charged cylinders

Note: In this section, we use V, instead of U, to denote the potential function.

Here we show how Laplace's equation, introduced earlier in this lectures, can be used to solve the potential function $V(\mathbf{r})$ that results from a given physical situation – one that you may have already investigated in your Physics labs, namely, two concentric charged cylinders with prescribed potentials. The top view of this system is sketched below. We assume that the radii of the cylinders are given by a and b. The potentials of these cylinders are V_a and V_b, respectively. At this time, we are not concerned whether V_a is greater than, less than or equal to V_b.



We wish to find the potential function $V(\mathbf{r})$ at a general point $\mathbf{r}$ between the two cylinders. Because of the cylindrical symmetry of the system, it is most convenient to use cylindrical coordinates (r, θ, z). The potentials on the cylinders are assumed to be constant over the entire cylinder, i.e., independent of z. As such we may assume that the potential V between the cylinders is independent of z, i.e., $V = V(r, \theta)$. A further simplification results because of the circular symmetry of the system – the potentials on the cylinders do not vary with θ. As a result, the potential V between the cylinders is independent of θ and therefore a function only of the radial coordinate r, i.e., $V = V(r)$. Indeed, we expect the level curves of V to be circles concentric with the two cylindrical sheets. This symmetry simplifies the problem greatly.

From the results stated just before this Appendix, Laplace's equation in cylindrical coordinates for a potential function $V(r, \theta, z)$ is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{75}$$

But recall that in this problem, V is independent of θ and z, which implies that the partial

derivatives involving these variables vanish. As a result, Laplace's equation for $V(r)$ becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0. \tag{76}$$

Note that we have replaced the partial derivatives by ordinary derivatives with respect to the single independent variable $V(r)$. The result is a second order linear differential equation in the unknown $V(r)$. This is the equation that we must solve. But recall that there are two conditions that the solution of this equation must satisfy, namely, the potentials at the cylinders, i.e.,

$$V(r_a) = V_a, \qquad V(r_b) = V_b. \tag{77}$$

Eq. (76) is actually rather easy to solve because of the absence of the function V from this equation. Let us set $W(r) = V'(r)$ so that Eq. (76) becomes

$$\frac{dW}{dr} + \frac{1}{r}W = 0. \tag{78}$$

This is a first order separable DE in $W(r)$. Separating variables, we obtain

$$\frac{dW}{W} = -\frac{dr}{r} \tag{79}$$

which is easily integrated to obtain

$$\ln W = -\ln r + C = \ln\left(\frac{1}{r}\right) + C, \tag{80}$$

where C is an arbitrary constant. Exponentiating both sides gives

$$W = \frac{dV}{dr} = \frac{A}{r}, \tag{81}$$

where $A = e^C$ is an arbitrary constant. We now solve for $V(r)$ by antidifferentiation:

$$V(r) = A\ln r + B, \tag{82}$$

where A and B are arbitrary constants whose values will be determined by the two "boundary conditions" in Eq. (77):

$$\begin{aligned} V(r_a) &= V_a = A\ln r_a + B \\ V(r_b) &= V_b = A\ln r_b + B \end{aligned} \tag{83}$$

Subtracting the first equation from the second yields

$$V_b - V_a = A\ln\left(\frac{r_b}{r_a}\right), \tag{84}$$

implying that

$$A = \frac{V_b - V_a}{\ln(r_b/r_a)}. \tag{85}$$

(Note that $r_b/r_a > 1$, implying that the logarithmic term is positive.) We may now use one of the two boundary value conditions, say the second one, to solve for B:

$$B = V_b - A \ln r_b = V_b - \frac{V_b - V_a}{\ln(r_b/r_a)}. \tag{86}$$

Substituting our results for A and B into the general solution, Eq. (82), we obtain

$$\begin{aligned} V &= \frac{V_b - V_a}{\ln(r_b/r_a)} \ln r + V_b - \frac{V_b - V_a}{\ln(r_b/r_a)} \\ &= \frac{V_b - V_a}{\ln(r_b/r_a)} \ln(r/r_b) + V_b. \end{aligned} \tag{87}$$

For convenience, we'll rewrite this result as

$$V(r) = \frac{V_a - V_b}{\ln(r_b/r_a)} \ln(r_b/r) + V_b, \quad r_a \le r \le r_b. \tag{88}$$

You can check that $V(r_a) = V_a$ and $V(r_b) = V_b$. We may also write this result as

$$V(r) = \frac{V_b - V_a}{\ln(r_b/r_a)} (\ln r - \ln r_b) + V_b, \quad r_a \le r \le r_b. \tag{89}$$

The most important consequence of this result is that the potential $V(r)$ does not vary linearly from V_a at $r = r_a$ to V_b at $r = r_b$, because of the appearance of the logarithmic function $\ln r$ in the equation. As a result, level curves of $V(r)$ of equal spacing ΔV, which will be circles in the xy-plane, will appear to be more "bunched up" near the inner circle, $r = r_a$. This is because the term $\ln r$ becomes steeper as r decreases. This "bunching" of level curves is sketched below.