Lecture 15

Multiple Integration

(Relevant section from Stewart, Section 15.1)

We now turn to the *integration* of scalar-valued functions $f : \mathbf{R}^n \to R$, i.e., $f(x_1, x_2, \cdots, c_n)$, over regions in $\mathbf{R}^n$. The need to perform such integrations is common in Physics. For example, we may wish to find:

1. The total charge Q in a region $R \subset \mathbf{R}^3$ that is enclosed by a closed surface S, e.g., a spherical surface of radius R centered at a point $\mathbf{p} \in \mathbf{R}^3$,

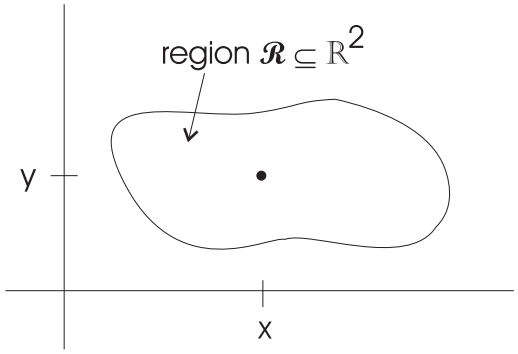
2. The probability of finding an electron in the $1s$ state of a hydrogen atom within the distance α/a_0 from the nucleus, where $\alpha \geq 0$ and a_0 is the so-called *Bohr radius*.

In all cases, the idea of integrating scalar-valued functions of several variables is a natural generalization of the integration of real-valued functions of a single variable, i.e., $f : \mathbf{R} \to \mathbf{R}$ over the real line $\mathbf{R}$. We simply have to keep in mind the "Spirit of Calculus," where we subdivide the region of interest and perform a summation over the subregions.

Double Integrals (Integration in $\mathbf{R}^2$)

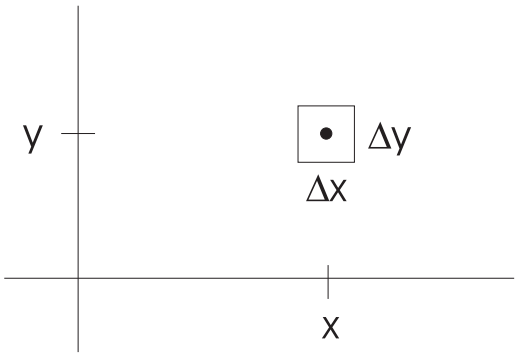
(Relevant sections from Stewart, Section 15.1-2)

We begin with the integration of functions of two variables, $f(x, y)$, over regions in the plane $\mathbf{R}^2$.



Suppose that a function $f : \mathbf{R} \to \mathbf{R^2}$ is defined over a region $R \subset \mathbf{R^2}$.

Example: $f(x, y)$ is the (2D or "areal") density of a thin plate at (x, y), a measure of the amount of mass per unit area around the point (x, y).



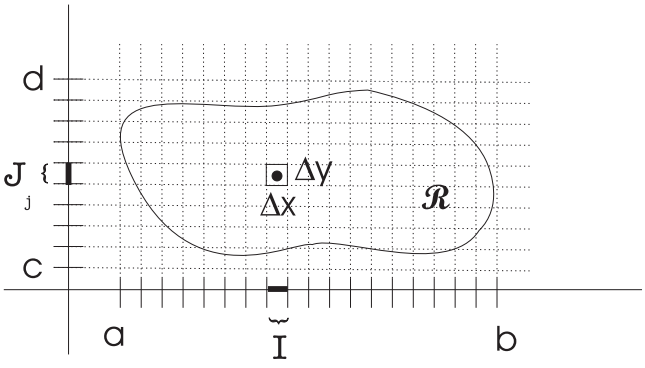
The amount of mass Δm in a small rectangle $\Delta x, \Delta y$ centered at (x, y) is

$$\Delta m \cong f(x, y)\Delta x, \Delta y$$

Question: What is the total mass of the plate?

Answer:

$$M = \int\int_R f(x, y)\, dA \qquad (\text{ where } dA = dx\, dy \text{ or } dy\, dx)$$



Suppose region R is contained in the rectangular region

$$a \leq x \leq b$$
$$c \leq y \leq d$$

In the "Spirit of Calculus," we perform the following procedures:

1. Divide $[a, b]$ into n subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$, of length $\Delta x = \frac{b-a}{n}$.

2. Divide $[c, d]$ into m subintervals $J_j = [x_{j-1}, x_j]$, $j = 1, 2, \cdots, m$, of length $\Delta y = \frac{d-c}{m}$.

   This produces $n \times m$ tiny rectangles of area $\Delta A = \Delta x \Delta y$.

3. From each rectangle, ΔA_{ij}, pick a sample point (x_i^*, y_j^*), where $x_i^* \in I_i, y_j^* \in J_j$.

4. Evaluate $f(x_i^*, y_j^*)$ at the sample point.

5. Now form the sum

$$S_{nm} \sum_{i=1}^{n}{}' \sum_{j=1}^{m}{}' f(x_i^*, y_j^*)\Delta A,$$

   where the primes indicate that we sum only over those rectangles ΔA_{ij} for which

$$\Delta A_{ij} \text{ contains points in } R.$$

Claim: In the limits $n, m \to \infty$, so that $\Delta A_{ij} = \Delta A \to 0$, the Riemann sums S_{mn}, i.e.,

$$S_{nm} \to \qquad \text{“} \int\int_R f(x,y)\, dA \qquad \text{”}$$

where the expression in quotes denotes the

“ double integral of the function f over the region R. ”

Particular applications:

1. If $f(x,y)$ is the mass density function ρ, then $\int\int_R \rho\, dA$ is the *total mass* of the plate.
2. If $f(x,y)$ is the charge density function ρ, then $\int\int_R \rho\, dA$ is the *total charge* on the plate.
3. For the special case where $f(x,y) = 1$, then $\int\int_R\ dA =$ *the area of* R, since we are simply summing over the areas ΔA of rectangles that contain points of R.

Practical Calculation:

We consider the summations involved in the Riemann sums as the result of a "nested 'for' loop":

1. **Column-wise Scan of R**

   `sum := 0`

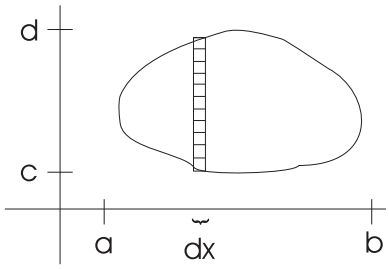
   `for` i `from` 1 `to` n $(a \leq x \leq b)$

   `for` j `from` $j_{lower}(i)$ `to` $j_{upper}(i)$ $(y_{lower}(x) \leq y \leq y_{upper}(x))$

   `sum := sum +` $f(\ )\Delta x \Delta y$

   `end for loop`

   `end for loop`



2. **Row-wise Scan of R**

   `sum:= 0`

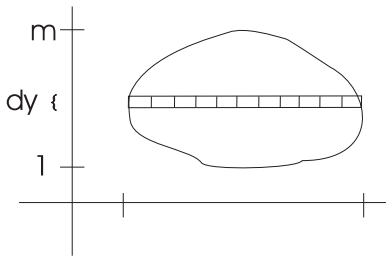
   `for` j `from` 1 `to` m $(c \leq x \leq d)$

   `for` i `from` $i_{lower}(j)$ `to` $i_{upper}(j)$ $(x_{lower}(y) \leq x \leq x_{upper}(y))$

   `sum := sum +` $f(\ )\Delta x \Delta y$

   `end for loop`

   `end for loop`



The simplest case is when the region R is a rectangular region, i.e., $a \leq x \leq b$, $c \leq y \leq d$, which we often write as the "tensor product", $[a, b] \times [c, d]$. In this case, we don't have to worry about what the "upper" or "lower" indices of integration are: We always integrate x from a to b and y from c to d. We illustrate with an example.

Example: Evaluate the double integral

$$\int \int_R (x^2 + 5y)\, dA, \tag{1}$$

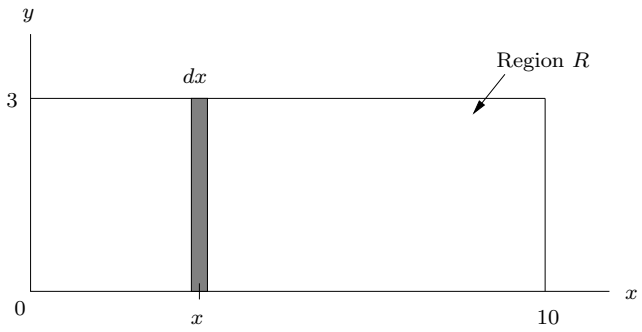
where R is the rectangular region, $0 \leq x \leq 10$, $0 \leq y \leq 3$.

Method No. 1: Columnwise scan

We scan the rectangular region with vertical slices of thickness dx centered at x. For each $x \in [0, 10]$, we scan the vertical slice from $y = 0$ to $y = 3$. As a result,

- **Outer integration:** $0 \leq x \leq 10$.
- **Inner integration:** $0 \leq y \leq 3$.

This is a nested integration - a continuous version of the columnwise nested summation that we presented earlier.



Columnwise integration over region R

We consider the inner integration first: For each $x \in [0, 10]$, we have the integral

$$\int_0^3 (x^2 + 5y)\, dy. \tag{2}$$

We "nest" this integral inside the outer integral over x. The result is

$$\int_0^{10} \int_0^3 (x^2 + 5y)\, dy\, dx. \tag{3}$$

This is known as an **iterated integral** (because we'll have to perform two integrations).

To evaluate this integral, we consider the inner integral in (2) first. Recall that x is considered to be fixed: For each fixed x, we integrate over y. We expect that the result will probably be dependent on x.

We use the Fundamental Theorem of Calculus to evaluate the inner integral. Since it involves an integration over y, we'll use the **partial antiderivative of the integrand with respect to** y, i.e.,

$$\int_0^3 (x^2 + 5y)\, dy = \left[x^2 y + \frac{1}{5}y^2\right]_{y=0}^{y=3}. \tag{4}$$

Note that I have written "$y = 0$" and "$y = 3$" for the limits, instead of simply "0" and "10". The latter is certainly acceptable, but when dealing with several variables, it is often helpful to indicate exactly which variable is going to be replaced with a value. In this case, it would be easy to replace x with 0 and 3.

Let us now go ahead and evaluate the above antiderivative:

$$\begin{aligned} \int_0^3 (x^2 + 5y)\, dy &= \left[x^2 y + \frac{1}{5}y^2\right]_{y=0}^{y=3} \\ &= 3x^2 + \frac{45}{2}. \end{aligned} \tag{5}$$

We now insert this result into the outer integral, i.e.,

$$\begin{aligned} \int_0^{10} 3x^2 + \frac{45}{2}\, dx &= \left[x^3 + \frac{45}{x}x\right]_0^{10} \\ &= 1000 + \frac{450}{2} \\ &= 1225. \end{aligned} \tag{6}$$

Therefore, the value of the integral over the rectangular region, as obtained by Method No. 1, is

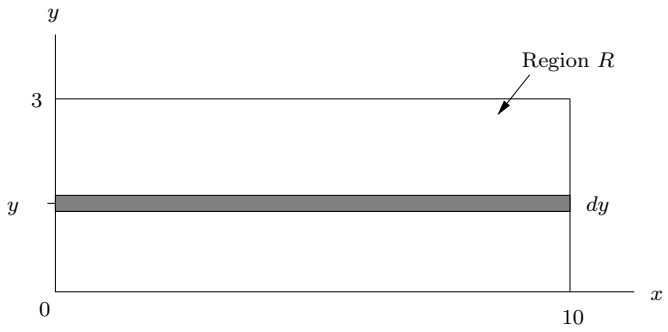
$$\int\int_R (x^2 + 5y)\, dA = 1225. \tag{7}$$

Method No. 2: Row-wise scan

We scan the rectangular region with horizontal slices of thickness dy centered at y. For each $y \in [0, 3]$, we scan the vertical slice from $x = 0$ to $x = 10$. As a result,

- **Outer integration:** $0 \leq y \leq 3$.
- **Inner integration:** $0 \leq x \leq 10$.

This is a continuous version of the second nested summation that we presented earlier.



Rowwise integration over region R

We consider the inner integration first: For each $y \in [0, 3]$, we have the integral

$$\int_0^{10} (x^2 + 5y)\, dx. \tag{8}$$

We "nest" this integral inside the outer integral over y. The result is

$$\int_0^3 \int_0^{10} (x^2 + 5y)\, dx\, dy. \tag{9}$$

Again, we consider the inner integral in (8) first. Recall that y is considered to be fixed: For each fixed y, we integrate over x. We expect that the result will probably be dependent on y.

Once again, we use the Fundamental Theorem of Calculus to evaluate the inner integral. Since it involves an integration over x, we'll use the **partial antiderivative of the integrand with respect to** x, i.e.,

$$\int_0^{10} (x^2 + 5y)\, dx = \left[\frac{1}{3}x^3 + 5xy\right]_{x=0}^{x=10}. \tag{10}$$

Now evaluate the above antiderivative:

$$\begin{aligned} \int_0^{10} (x^2 + 5y)\, dx &= \left[\frac{1}{3}x^3 + 5xy\right]_{x=0}^{x=10} \\ &= \frac{1000}{3} + 50y. \end{aligned} \tag{11}$$

We now insert this result into the outer integral, i.e.,

$$\begin{aligned}\int_0^3 \left(\frac{1000}{3} + 50y\right)\, dy &= \left[\frac{1000}{3}y + 25y^2\right]_0^3 \\ &= 1000 + 225 \\ &= 1225. \end{aligned} \tag{12}$$

Therefore, using Method No. 2, the value of our integral over the rectangular region is

$$\int\int_R (x^2 + 5y)\, dA = 1225, \tag{13}$$

in agreement with that of Method No. 1.

This example illustrates that, as probably expected, the value of the integral does not depend on the method used to evaluate it. In fact, for sufficiently "nice" functions and boundary curves, the value of the double integral,

$$\int\int_D f(x, y)\, dA, \tag{14}$$

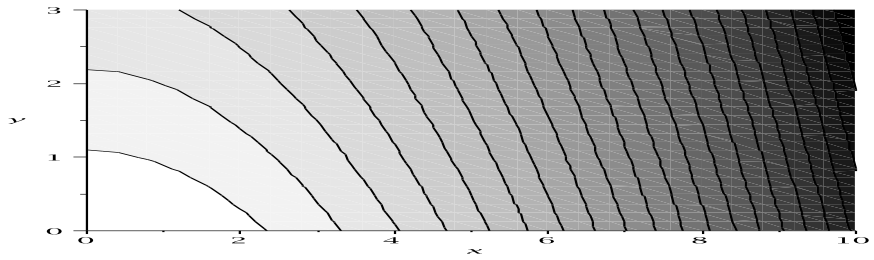
should be independent of the method of integration used. What does "nice" mean? Well, for example,

1. $f(x, y)$ is continuous over the region D and
2. the boundary ∂D of D is piecewise differentiable, that is, it is the union of a finite number of curves C_i, each of which have well-defined tangent vectors at all points, except possibly at the endpoints.

That being said, we may relax the requirement in 1) to read $f(x, y)$ is piecewise continuous over the region D. The formal mathematical result is known as **Fubini's Theorem**. You can find a short discussion in Stewart's textbook, p. 961.

The "meaning" of this double integral

Think of the integrand $f(x, y) = x^2 + 5y$ as a weighting function over the region of integration R – it could represent the mass density of a rectangular plate. At $(0, 0)$ the density is 0. At $(10, 3)$ the density is 115. The plate is getting denser as we move away from the origin, particularly in the x-direction. The contour plot of the integrand $f(x, y) = x^2 + 5y$ shown below may help. The value of



Contour plot of integrand $f(x, y) = x^2 + 5y$ over rectangular region R

the double integral could be the total mass M of the plate. Later, we'll compute the center of mass of this plate, and see the effects of the nonuniform distribution of its mass.

Lecture 16

Double integrals (cont'd)

(Relevant sections from Stewart, Section 15.1-2)

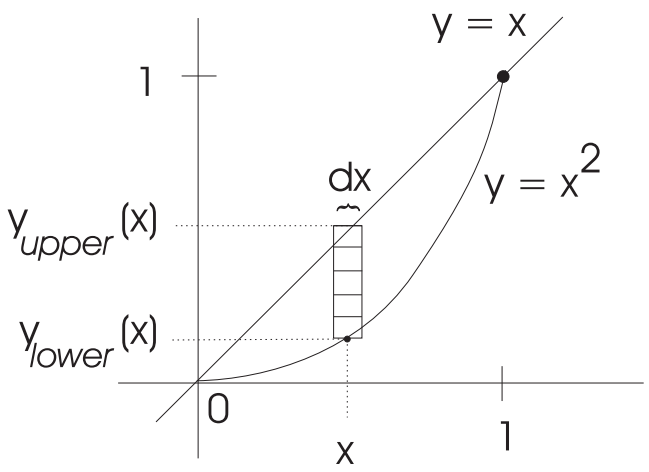
In the previous lecture, we introduced the idea of a double integral and illustrated it with an example that involved a simple integration over a rectangular region. We now consider a slightly more complicated, nonrectangular region.

Example: Evaluate the double integral,

$$\int\int_R (x^2 + y^2)\, dA$$

where R is the region in the first quadrant contained by the curves $y = x$ and $y = x^2$, $0 \le x \le 1$.

Method No. 1: Columnwise Scan



Scan a vertical slice from x^2 to x running horizontally.

Outer loop: $0 \le x \le 1$

Inner loop: $\underbrace{x^2}_{y_{lower}} \le y \le \underbrace{x}_{y_{upper}}$

This nesting produces the following iterated integral:

$$\int_0^1 \int_{x^2}^{x} (x^2 + y^2)\, dy. \tag{15}$$

As before, we perform the inner integration first, i.e.,

$$\int_{x^2}^{x} (x^2 + y^2)\, dy. \tag{16}$$

Keeping x fixed and integrating partially w.r.t. y, we obtain,

$$\int_{x^2}^{x} (x^2 + y^2)\, dy \;=\; \left[x^2 y + \frac{1}{3}y^3\right]_{y=x^2}^{y=x}$$

$$
\begin{aligned}
&= \left(x^3 + \frac{1}{3}x^3\right) - \left(x^4 + \frac{1}{3}x^6\right) \\
&= \frac{4}{3}x^3 - x^4 - \frac{1}{3}x^6
\end{aligned}
$$

Now perform the **outer** integration by inserting this result:

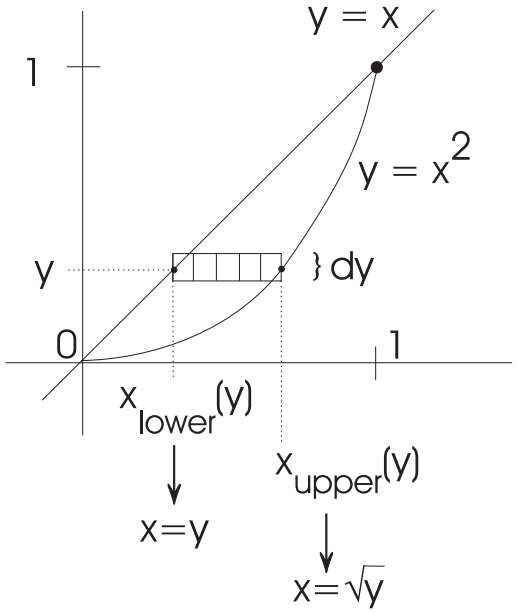
$$
\begin{aligned}
\Rightarrow \int_0^1 \frac{4}{3}x^3 - x^4 - \frac{1}{3}x^6 \, dx &= \frac{1}{3}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \Bigg|_0^1 \\
&= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} \\
&= \frac{35 - 21 - 5}{105} \\
&= \frac{9}{105} \\
&= \frac{3}{35}
\end{aligned}
$$

Therefore, we find that

$$\int\int_R (x^2 + y^2) \, dA = \frac{3}{35}$$

for the columnwise method.

Method No. 2: Rowwise Scan



Scan a horizontal slice from y to $\sqrt{y}$ running vertically.

Outer loop: $0 \leq y \leq 1$

Inner loop: $\underbrace{y}_{x_{lower}} \leq x \leq \underbrace{y^{\frac{1}{2}}}_{x_{upper}}$

The resulting iterated integral is

$$\int_0^1 \int_y^{y^{1/2}} (x^2 + y^2)\, dx\, dy. \tag{17}$$

We now perform the inner integration over x first:

$$\int_y^{y^{1/2}} (x^2 + y^2)\, dx.$$

In this case, we treat y as a constant and partially integrate with respect to x. I.e., we are keeping y fixed as we integrate over x.

$$\begin{aligned}
\int_y^{y^{1/2}} (x^2 + y^2)\, dx &= \left[\frac{1}{3}x^3 + xy^2\right]_{x=y}^{x=y^{1/2}} \\
&= \left(\frac{1}{3}y^{3/2} + y^{5/2}\right) - \left(\frac{1}{3}y^3 + y^3\right) \\
&= \frac{1}{3}y^{3/2} + y^{5/2} - \frac{4}{3}y^3
\end{aligned}$$

Now insert this result into the **outer** integration:

$$\begin{aligned}
\int_0^1 \frac{1}{3}y^{3/2} + y^{5/2} - \frac{4}{3}y^3\, dx &= \left[\frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{1}{3}y^4\right]_0^1 \\
&= \frac{2}{15} + \frac{2}{7} - \frac{1}{3} \\
&= \frac{14 + 30 - 35}{105} \\
&= \frac{3}{35}
\end{aligned}$$

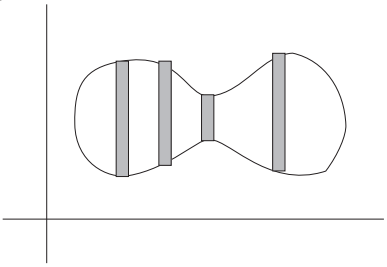
Thus, we have obtained the same result for both rowwise and columnwise methods – as hoped!

Question:

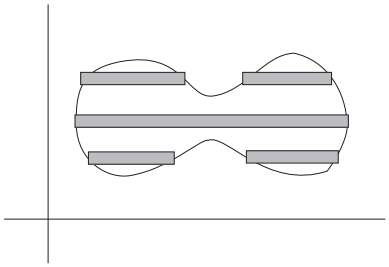
In which order should you perform the double integration?

Answer:

Use whatever works! In some cases, one method is better, practically speaking, than another. E.g., consider the case of the following "dumb bell" shaped regions:



is probably easier to use than

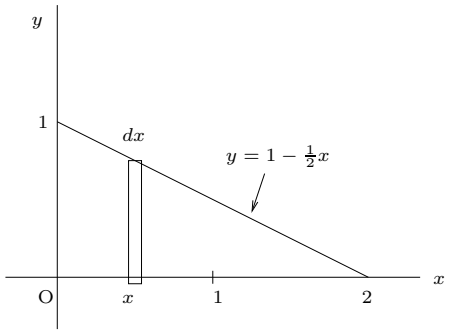


The most important point to remember is that, whatever method you use, *it is the boundary curve(s) of the region D that determine(s) the limits of integration in the double integral.*

Example: (This example was not done in class.) Evaluate $\int\int_D xy\ dA$, where D is the triangular region in the first quadrant with vertices (0,0), (2,0) and (0,1).

The boundary of the region consists of the x and y-axes and the line connecting the vertices (0,1) and (2,0), which is given by the equation $y = 1 - x/2$.

Method 1: Columnwise integration. The outer integration is performed for $0 \le x \le 2$.



For each $x \in [0, 2]$, we integrate y from the x-axis, i.e. $y = 0$ up to the line $y = 1 - x/2$. Thus

$$\int\int_D xy\ dA = \int_0^2 \int_0^{1-x/2} xy\ dy\ dx \tag{18}$$

The inner integral is

$$\left[\frac{1}{2}xy^2\right]_0^{1-x/2} = \frac{1}{2}x\left(1 - \frac{1}{2}x\right)^2. \tag{19}$$

Insertion into the outer integral gives

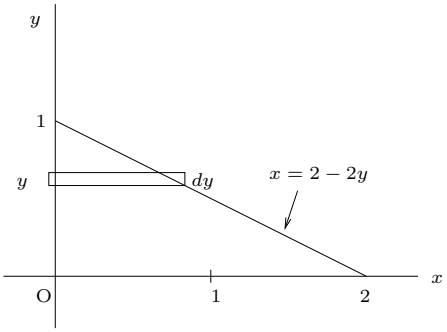
$$\int_0^2 \left(\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{8}x^3\right) dx = \cdots = \frac{1}{6}. \tag{20}$$

Method 2: Row-wise integration. The outer integration is performed for $0 \le y \le 1$. For each $y \in [0, 1]$, we integrate x from the y-axis, i.e., $x = 0$ to the line $y = 1 - x/2$. But we must solve for x in terms of y on this line:

$$y = 1 - \frac{x}{2} \text{ implies } x = 2 - 2y.$$

Thus

$$\int\int_D xy\ dA = \int_0^1 \int_0^{2-2y} xy\ dx\ dy = \cdots = \frac{1}{6}. \tag{21}$$



Sometimes, although the region D appears amenable to integration using both methods, it may not be possible to use one, or perhaps even both, methods.

Example: $\int\int_D \sin\left(\frac{x}{y}\right)\, dA$, where D is the region in the first quadrant enclosed by the y-axis, the line $y = \pi$ and the curve $y = \sqrt{x}$.

Columnwise integration yields the following double integral:

$$\int_0^{\pi^2} \int_{\sqrt{x}}^{\pi} \sin\left(\frac{x}{y}\right) dydx. \tag{22}$$

It is not possible to antidifferentiate the inner integral in closed form.

Row-wise integration yields the double integral

$$\int_0^{\pi} \int_0^{y^2} \sin\left(\frac{x}{y}\right) dxdy. \tag{23}$$

The inner integral may be evaluated:

$$\begin{aligned} \int_0^{y^2} \sin\left(\frac{x}{y}\right) dx &= \left[-y\cos\left(\frac{x}{y}\right)\right]_{x=0}^{x=y^2} \\ &= -y\cos y + y\cos 0 \\ &= -y\cos y + y \end{aligned} \tag{24}$$

Insertion into the outer integral yields

$$\begin{aligned} \int_0^{\pi} (-y\cos y + y)dy &= \left[\cos y - y\sin y + \frac{1}{2}y^2\right]_0^{\pi} \\ &= \frac{1}{2}\pi^2 + 2. \end{aligned} \tag{25}$$

Some properties of double integrals

(Relevant section from Stewart, Section 15.3, pp. 970-971)

Here we list some important properties of double integrals that follow from the linearity of the integration operator. (Recall that integration is a limiting operation involving the linear "summing" up of "pieces" of an integrand.) These properties are two-dimensional versions of properties that you saw for definite integrals of functions of a single real variable. In what follows, we assume that all functions are defined over the regions of integration, and that the definite integrals exist. (Following Stewart, we denote the regions as D, etc..)

1. Additivity of integrand,

$$\int\int_D [f(x,y) + g(x,y)]\, dA = \int\int_D f(x,y)\, dA + \int\int_D f(x,y)\, dA. \tag{26}$$

2. Scalar multiplication: For any constant $c \in \mathbf{R}$,

$$\int\int_D cf(x,y)\, dA = c \int\int_D f(x,y)\, dA. \tag{27}$$

3. If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then

$$\int\int_D f(x,y)\, dA \leq \int\int_D g(x,y)\, dA. \tag{28}$$

4. As mentioned earlier, in the special case $f(x,y) = 1$ for all $(x,y) \in D$, we have

$$\int\int_D f(x,y)\, dA = \int\int_D 1\, dA = A(D), \quad \text{the area of } D. \tag{29}$$

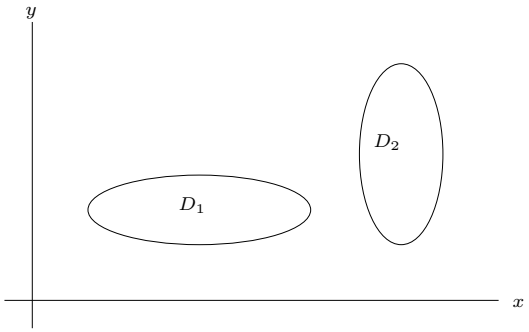
5. We can combine 2, 3, and 4 to yield the following result: If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$mA(D) \leq \int\int_D f(x,y)\, dA \leq MA(D). \tag{30}$$

   This result is useful in applications where estimates of integrals are required.

The first two properties listed above are quite straightforward – indeed, one would simply make use of them automatically. They, along with the other properties, can be proved by means of the Riemann sum definition of the double integral.

Here is another property that one may automatically take for granted: Suppose that D_1 and D_2 are disjoint regions in the plane, i.e., $D_1 \cap D_2 = \phi$ (empty set), as sketched below.

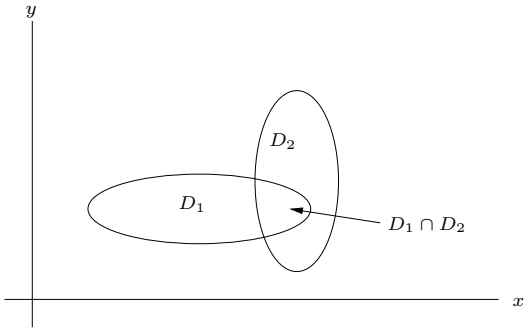


Then

$$\int\int_{D_1 \cup D_2} f(x,y)\, dA = \int\int_{D_1} f(x,y)\, dA + \int\int_{D_2} f(x,y)\, dA. \tag{31}$$

This seems quite obvious.

If, on the other hand, the two regions overlap, as sketched below, then the RHS of the above equation



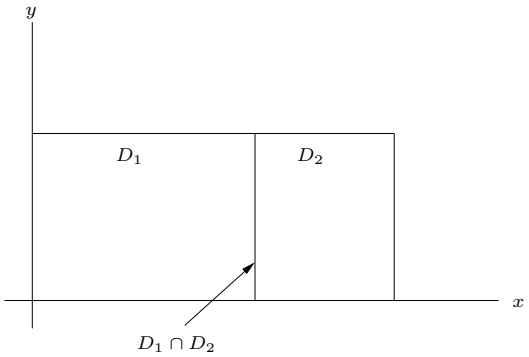
would actually integrate over the region of overlap $D_1 \cap D_2$ twice. As such, we would have to subtract one such contribution, i.e.,

$$\int\int_{D_1 \cup D_2} f(x,y)\, dA = \int\int_{D_1} f(x,y)\, dA + \int\int_{D_2} f(x,y)\, dA - \int\int_{D_1 \cap D_2} f(x,y)\, dA. \tag{32}$$

But we rarely have to worry about such situations.

What one does encounter quite often, however, is the following situation: The sets D_1 and D_2 share a common boundary, or piece of one. In other words, the set $D_1 \cap D_2$ is a **curve** in $\mathbf{R}^2$, as sketched below.

In this situation, the final term on the RHS in Eq. (32) is an integration over a **curve**. Since the double integral is based on summations over **areas** (yes, OK, they are infinitesimal areas in the



limit), and the **area of a curve in $\mathbf{R}^2$ is zero**, then it follows that the integration over the curve is zero. As such, we may proceed as if the regions D_1 and D_2 are disjoint. (In mathematical parlance, a curve has "zero two-dimensional Lebesgue measure", a fancy way of saying that it has zero area.)

Integration over circular regions

Integrations over circular regions are common in Physics. As an example, we consider the integral of a function $f(x, y)$ over the circular region D defined by $x^2 + y^2 \leq R^2$, i.e.,

$$\int\int_D f(x, y)\, dA. \tag{33}$$

Planar polar coordinates are ideally suited to treat such problems, and we shall discuss integration of functions in terms this coordinate system a little later. Even though Cartesian coordinates are not well-suited for such problems and may require much work, it still helps to be prepared to address them in this way.

If we perform a columnwise integration, then the outer integration variable x will range from $-R$ to R. For each $x \in [-R, R]$, the integration variable y will range from the lower semicircle to the upper semicircle, i.e. $-\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$. Thus the above integral will become the iterated integral

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f(x, y)\, dy\, dx. \tag{34}$$

For example, the area of this circular region is obtained by setting $f(x, y) = 1$, i.e.,

$$A = \int\int_D dA = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy\, dx. \tag{35}$$

The inner integral is simply

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = 2\sqrt{R^2 - x^2}. \tag{36}$$

Inserting this result into the inner integral gives

$$A = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx. \tag{37}$$

This is the integral that you would have obtained in first-year calculus to compute the area between the curves $\sqrt{R^2 - x^2}$ and $-\sqrt{R^2 - x^2}$, hence the area of the circle of radius R.

Just to refresh our memories, it is convenient to introduce the trigonometric substitution

$$x = R\sin\theta. \tag{38}$$

This substitution will elminate the square root sign from the integrand. Since x must range from $-R$ to R, it follows that $\sin\theta$ must range from -1 to 1, implying that $-\pi/2 \le \theta \le \pi/2$.

We still have to take care of the infinitesimal element dx. Taking differentials of (38), we have

$$dx = Rd(\sin\theta) = R\cos\theta d\theta. \tag{39}$$

Then

$$\begin{aligned} 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx &= 2R^2 \int_{-\pi/2}^{\pi/2} R\cos\theta R\cos\theta d\theta \\ &= 2R^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta \\ &= 2R^2 \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= 2R^2 \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} \\ &= 2R^2 \frac{\pi}{2} \\ &= \pi R^2. \end{aligned} \tag{40}$$

Once we have established the limits of integration we can, in principle, integrate any function $f(x, y)$. For example, if $f(x, y) = x^2 + y^2$, then our integral would be

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (x^2 + y^2)\, dy\, dx. \tag{41}$$

Inner integral:

$$\begin{aligned} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (x^2 + y^2)\, dy &= \left[x^2 y + \frac{1}{3}y^3\right]_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} \\ &= 2x^2(R^2 - x^2)^{1/2} + \frac{2}{3}(R^2 - x^2)^{3/2}. \end{aligned} \tag{42}$$

Inserting this result into the outer integral gives

$$\int_{-R}^{R} \left[2x^2 (R^2 - x^2)^{1/2} + \frac{2}{3}(R^2 - x^2)^{3/2} \right] \, dx \;=\; \frac{\pi R^4}{2}. \tag{43}$$

We leave the final result as an exercise.

Lecture 17

Double Integrals (cont'd)

Infinite domains and improper double integrals

It is often the case that one, or both, of the ranges of integration are infinite. This is not a problem – we simply treat such integrals as improper integrals, as was done for functions of a single variable.

Example: Find $\int\int_D e^{-(x+y)}\,dA$, where D is the semi-infinite strip $x \geq 0$, $0 \leq y \leq 1$.
If we employ columnwise integration, the iterated double integral is

$$I = \int_0^\infty \int_0^1 e^{-(x+y)}\,dy\,dx, \tag{44}$$

which may be written as

$$I = \int_0^\infty e^{-x} \int_0^1 e^{-y}\,dy\,dx, \tag{45}$$

where we have taken the factor e^{-x}, which is independent of y, out of the inner integral.

The inner integral becomes

$$\int_0^1 e^{-y}\,dy = \left[-e^{-y}\right]_0^1 = 1 - e^{-1}. \tag{46}$$

This factor is independent of x and becomes a multiplicative factor in (45):

$$I = (1 - e^{-1}) \int_0^\infty e^{-x}\,dx. \tag{47}$$

The integral is an improper one, understood as

$$\begin{aligned} \int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty} \int_0^b e^{-x}\,dx \\ &= \lim_{b\to\infty} \left[-e^{-x}\right]_0^b \\ &= \lim_{b\to\infty} \left[1 - e^{-b}\right] \\ &= 1. \end{aligned} \tag{48}$$

The net result is

$$I = 1 - e^{-1}. \tag{49}$$

Rowwise integration produces the same result, as expected.

A comment regarding the integral in (45): The integrand $e^{-(x+y)}$ in this problem is a **separable** integrand – it can be expressed as a product of a purely x-dependent function and a purely y-independent function.

$$e^{-(x+y)} = e^{-x}e^{-y}. \tag{50}$$

Because of this separability, and the fact that *the limits of integration are not dependent upon* x or y, the integral factors as a product of two definite integrals – one over x and the other over y:

$$\begin{aligned} I &= \int_0^\infty e^{-x} \int_0^1 e^{-y}\, dy\, dx \\ &= \left(\int_0^\infty e^{-x}\, dx\right)\left(\int_0^1 e^{-y}\, dx\right) \\ &= (1)(1-e^{-1}). \end{aligned} \tag{51}$$

In general, if an integrand $f(x,y)$ is **separable**, i.e., may be written as the product $p(x)q(y)$, and the limits of integration are constants, then

$$\int_a^b \int_c^d f(x,y)\, dx\, dy = \int_a^b \int_c^d p(x)q(y)\, dx\, dy = \left(\int_a^b p(x)\, dx\right)\left(\int_c^d q(x)\, dy\right). \tag{52}$$

Separable integrals are often encountered in applications, as we shall even see in this course. In many problems involving spherical symmetry, it is convenient to separate out and integrate over the angular variables first, leaving an integration over the single radial variable.

Warning: That being said, one must exercise care, making sure that the procedure of separating variables is indeed justified.

Some physical applications

Mean value of a function

Recall the definition of the **mean value** (or **average value**) of the function $f : \mathbf{R} \to \mathbf{R}$ over the interval $[a,b]$:

$$\bar{f}_{[a,b]} = \frac{\int_a^b f(x)\, dx}{\int_a^b\, dx} = \frac{\int_a^b f(x)\, dx}{(b-a)}. \tag{53}$$

The mean value of a function $f(x,y)$ over a bounded region $D \subset \mathbf{R}^2$ is defined in an analogous way:

$$\bar{f}_D = \frac{\int\int_D f(x,y)\, dA}{\int\int_D\, dA} = \frac{\int\int_D f(x,y)\, dA}{A(D)}, \tag{54}$$

where $A(D)$ denotes the area of region D.

If $f(x,y)$ is continuous on a finite region $D \subset \mathbf{R}^2$, then it is bounded on D, that is, there exist $m, M \in \mathbf{R}$ such that

$$m \leq f(x,y) \leq M, \quad \text{for all} \quad x,y \in D. \tag{55}$$

Because of the continuity of $f(x,y)$ on D, there exists at least one point, say $(x_m, y_m) \in D$ at which $f(x,y)$ achieves its minimum value m and one point, say $(x_M, y_M) \in D$ at which $f(x,y)$ achieves its maximum value M. Furthermore, the continuity of $f(x,y)$ over D guarantees that there exists at least one point $(a,b) \in D$ at which $f(x,y)$ achieves its mean value, i.e.,

$$f(a,b) = \bar{f}_D. \tag{56}$$

Center of mass of a thin plate

Suppose that a finite region $D \subset \mathbf{R}^2$ represented a thin metal plate and that the *areal* mass density of the plate (mass/unit area) at a point $(x,y) \in D$ is given by $\rho(x,y)$. Then the infinitesimal element of mass dm at (x,y) is given by

$$dm = \rho(x,y)\, dA. \tag{57}$$

At this point, it doesn't hurt to examine what this mass density function $\rho(x,y)$ represents. For the same idea will apply to charge densities in two and three dimensions. The "density function" $\rho(x,y)$ at a point is a mathematical abstraction or approximation. You could define it as

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A} = \lim_{\epsilon \to 0} \frac{\Delta m}{\epsilon^2}. \tag{58}$$

where Δm is the amount of mass contained in a two-dimensional box with sides of length radius ϵ. (We could also have used a circular region of radius ϵ and area $\Delta A = \pi\epsilon^2$. This would only have changed $\rho(x,y)$ by a factor of π.) Note that in this definition, we are looking at the plate from above and ignoring its thickness. In reality, it is quite possible that the thickness will vary as we move from point to point. But this is taken into consideration in our definition of $\rho(x,y)$, since we consider all of the material of the plate that lies in the 2D box as we look at the plate from above.

The definition of the density function $\rho(x,y)$ is actually based on a *continuum* approximation, where we treat materials as continuous bodies that can be divided up into arbitrarily small pieces. Of course, in the real world, this procedure is meaningless because of the atomic nature of matter. Nevertheless it is a very good approximation in the macroscopic treatment of materials. We are essentially "smearing" out the concentrations of mass at nuclei, molecules, etc., to produce a kind of

"lower-resolution" picture of matter, in order to be able to employ the techniques of calculus, which is formulated over continuous variables. The same procedure is done with charges and charge densities. A rearrangement of (58) gives $\Delta m = \rho(x, y)\Delta A$ which, in the "infinitesimal limit," becomes (57).

OK, back to Eq. (57). If we now integrate these mass elements over the entire region D, we obtain the total mass M of the plate:

$$M = \int\int_D dm = \int\int_D \rho \, dA. \tag{59}$$

The coordinates $(\bar{x}, \bar{y})$ of center of mass of the plate are given by

$$\bar{x} = \frac{\int\int_D x\rho \, dA}{M}, \qquad \bar{y} = \frac{\int\int_D y\rho \, dA}{M}. \tag{60}$$

These equations represent a continuum version of the discrete case, where we have N masses m_k, $1 \leq k \leq N$, situated at points (x_k, y_k). In this case, the center of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{1}{M}\sum_{k=1}^{N} m_i x_i, \qquad \bar{y} = \frac{1}{M}\sum_{k=1}^{N} m_i y_i, \tag{61}$$

where

$$M = \sum_{k=1}^{N} m_k \tag{62}$$

is the total mass of the system.

In the special case that $\rho(x, y) = \rho_0$, a constant, then these constants can be taken out of all double integrals in the numerators and denominators of Eq. (60). Furthermore, their appearance in the numerator and denominator allows them to be cancelled out, and the center of mass becomes the *centroid* of region D.

Example 1: We return to the first double integral problem examined in this section (Lecture 15) on double integrals, namely the integration of the function $f(x, y) = x^2 + 5y$ over the rectangular region D defined by $0 \leq x \leq 10$ and $0 \leq y \leq 3$. We found that

$$\int\int_D (x^2 + 5y) \, dA = 1225. \tag{63}$$

If we now interpret this function as the mass density function $\rho(x, y) = x^2 + 5y$ of a thin plate that occupies region D, we have that the total mass of the plate is

$$M = \int\int_D dm = \int\int_D \rho \, dA = 1225. \tag{64}$$

Let us now determine the center of mass of this plate. We have to compute the additional integrals,

$$\begin{aligned} \int\int_D x\, dm &= \int\int_D x\rho\, dA = \int_0^{10}\int_0^3 x(x^2+5y)\, dydx \\ \int\int_D y\, dm &= \int\int_D y\rho\, dA = \int_0^{10}\int_0^3 y(x^2+5y)\, dydx. \end{aligned} \tag{65}$$

To compute the first integral, we must start with the inner integral,

$$\begin{aligned} \int_0^3 x(x^2+5y)\, dy &= \int_0^3 (x^3+5xy)\, dy \\ &= \left[x^3y + \frac{5}{2}xy^2\right]_0^3 \\ &= 3x^3 + \frac{45}{2}x \end{aligned} \tag{66}$$

Now insert this result into the outer integral,

$$\begin{aligned} \int_0^{10} \left(3x^3 + \frac{45}{2}x\right)\, dx &= \left[\frac{3}{4}x^4 + \frac{45}{4}x^2\right]_0^{10} \\ &= \frac{30000}{4} + \frac{4500}{4} \\ &= 8625. \end{aligned} \tag{67}$$

To compute the second integral, we start with the inner integral

$$\begin{aligned} \int_0^3 x(x^2+5y)\, dydx &= \left[\frac{1}{2}x^2y^2 + \frac{5}{3}y^3\right]_0^3 \\ &= \frac{9}{2}x^2 + \frac{45}{3}. \end{aligned} \tag{68}$$

Now insert this result into the outer integral,

$$\begin{aligned} \int_0^{10} \frac{9}{2}x^2 + \frac{45}{3}\, dx &= \left[\frac{3}{2}x^3 + \frac{135}{3}x\right]_0^{10} \\ &= 1500 + 450 \\ &= 1950. \end{aligned} \tag{69}$$

We now employ these result in the expressions in (60) to obtain

$$\bar{x} = \frac{8625}{1225} \approx 7.04; \qquad \bar{y} = \frac{1950}{1225} \approx 1.59. \tag{70}$$

The center of mass of this plate is roughly at $(7.04, 1.59)$. This result seems to make sense: Recall that the density function $\rho(x,y) = x^2 + 5y$ is small near the y-axis (and zero at the origin) and increases

as we move to the right as well as upward. One would then expect that the center of mass is shifted to the right and upward from the *centroid* of the plate, which is at $(5, 1.5)$.

Example 2: We return to the second problem examined earlier, namely, the integration of the function $x^2 + y^2$ over the region D in the first quadrant enclosed by the curves $y = x$ and $y = x^2$. We found that

$$\int\int_D (x^2 + y^2)\, dA = \frac{3}{35}. \tag{71}$$

Let us once again interpret this function as the mass density function $\rho(x, y) = x^2 + y^2$ of a thin plate that occupies region D so that the total mass of the plate is

$$M = \int\int_D \rho\, dm = \frac{3}{35}. \tag{72}$$

Let us now determine the center of mass of this plate. We find that (Exercise):

$$\begin{aligned} \int\int_D x\, dm &= \int\int_D \rho\, dA = \int_0^1 \int_{x^2}^x x(x^2 + y^2)\, dydx = \frac{7}{120}, \\ \int\int_D y\, dm &= \int\int_D \rho\, dA = \int_0^1 \int_{x^2}^x y(x^2 + y^2)\, dydx = \frac{16}{315}, \end{aligned} \tag{73}$$

so that

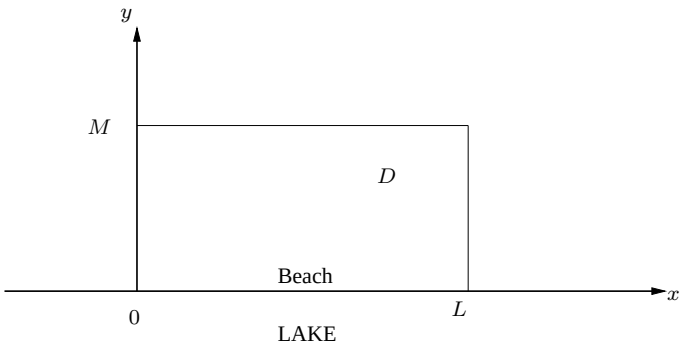
$$\bar{x} = \frac{7}{120} \cdot \frac{35}{3} = \frac{49}{72} \approx 0.68, \qquad \bar{y} = \frac{16}{315} \cdot \frac{35}{3} = \frac{16}{27} \approx 0.59. \tag{74}$$

The coordinate of the center of mass is $(\bar{x}, \bar{y}) \approx (0.68, 0.59)$, which seems to make sense. The density function $\rho = x^2 + y^2$ is small near the origin (in fact, zero at the origin) and increases as one moves away from $(0, 0)$ to $(1, 1)$. One would expect that the center of mass is shifted to the right of the *centroid* of the plate. (Exercise: Compute the centroid.)

Appendix: An interesting application of double integrals

At the end of Lecture 16, a student came with a very interesting application of double integration, following up on my comments about uneven land values. Consider a situation involving lakefront property: A parcel of land D borders a lake, as sketched below.

Since having property near a lake is extremely desirable, it doesn't seem unreasonable to assume that the value of a small piece of land, which could be very high right at the lake, decreases as we move away from the lake. In other words, from the above diagram, the price per unit area is given by



$f(x, y)$, which decreases as we increase y. The simplest model for such a decrease is a linear one of the form,

$$f(x, y) = S - By, \qquad V, B > 0. \tag{75}$$

Here S represents the value of land (per unit area) at the shore and B is the rate of decrease (per unit length) of the land. The value of a small piece of land of area $\Delta A = \Delta x \Delta y$ and centered at the point (x, y) is roughly

$$\Delta V \approx f(x, y)\Delta A. \tag{76}$$

The total value V of the property D sketched in the figure, where $0 \leq x \leq L$ and $0 \leq y \leq M$, is

$$\begin{aligned} V &= \int\int_D f(x, y)\, dA \\ &= \int_0^L \int_0^M (S - By)\, dy\, dx \\ &= SLM - \frac{1}{2}BLM^2 \\ &= SA(D)\left(1 - \frac{BM}{2S}\right), \end{aligned} \tag{77}$$

where $A(D) = LM$ denotes the area of the property. Note that if $B = 0$, meaning that all of the land is valued at the beachfront rate A, then the total value of the property is simply $SA(D)$, as expected.