Lecture 27

Line integrals: Integration along curves in $\mathbf{R}^n$

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.2, pp. 1034-1041)

In this section, we shall be integrating scalar-valued functions $f(x, y, z)$ and vector-valued functions $\mathbf{F}(x, y, z)$ along *curves* in $\mathbf{R}^3$. The motivation for doing these things will become clear as the discussion progresses.

In what follows, it will be important to keep in mind that we shall be working on curves C in $\mathbf{R}^3$ that will be parametrized in terms of a single variable as follows, Thus

$$\mathbf{x}(t) = \mathbf{r}(t) = (x(t), y(t), z(t)), \qquad a \leq t \leq b. \tag{1}$$

The notation "$\mathbf{r}(t)$" conforms to the notation used in the textbook. In most applications, the curves will be either

1. "C^1 paths," meaning that the components have continuous derivatives for all $t \in [a, b]$.

   Example: The curve $\mathbf{r}(t) = (x(t), y(t), z(t)) = (\cos t, \sin t, 2t)$ for $t \in [0, \pi]$

2. "Piecewise C^1 paths," meaning that the interval $[a, b]$ can be broken down into a union of subintervals $[t_{k-1}, t_k]$, $k = 1, 2, \cdots, n$, where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, and the path $\mathbf{r}(t)$ is C^1 on each subinterval. (The derivatives may not necessarily exist at the points t_k.

   Example: The square in $\mathbf{R}^2$ with corners (0,0), (1,0), (1,1) and (0,1), parametrized as

$$\mathbf{r}(t) = (x(t), y(t)) = \begin{cases} (t, 0), & t \in [0, 1], \\ (1, t-1), & t \in [1, 2], \\ (3-t, 1), & t \in [2, 3], \\ (0, 4-t), & t \in [3, 4], \end{cases} \tag{2}$$

Let us briefly recall the working definition of the Riemannian integral of the single-variable function $f(x)$ over the interval $[a, b]$:

1. We partition the interval $I = [a, b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$ using the partition points $x_k = a + k\Delta x$, $k = 0, 1, 2, \cdots, n$.

2. From each subinterval $I_k = [x_{k-1}, x_k]$, pick a sample point x_k^* and evaluate $f(x_k^*)$.

3. Form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Now let $n \to \infty$, implying that $\Delta x \to 0$. If $f(x)$ is sufficiently "nice" (i.e., piecewise continuous), then

$$\lim_{n\to\infty} S_n = S = \int_a^b f(x)\, dx$$

Some simple applications of the Riemann integral to physics

(This section was not covered in the lecture, but has been added for important review.)

1. **Total mass of a wire:** The interval $[a, b]$ could represent a very thin wire. The function $f(x)$ could represent the *lineal mass density* along the wire, the amount of mass per unit length at a point $x \in [a, b]$. (The contribution over the (small) cross sectional area is taken into account in the construction of f.) In the above procedure, the partitioning of the interval I into subintervals I_k breaks up the wire into n pieces of length Δx. The mass of each piece Δm_k is approximated as

$$\Delta m_k \approx f(x_k^*)\Delta x.$$

In other words, we are making the approximation that the lineal density $f(x) \approx f(x_k^*)$ over the interval $I_k = [x_{k-1}, x_k]$. The total mass of the wire is then

$$M = \sum_{k=1}^{n} \Delta m_k \approx \sum_{k=1}^{n} f(x_k^*)\Delta x = S_n.$$

In the limit $n \to \infty$, the total mass of the wire becomes

$$M = \int_a^b f(x)\, dx.$$

Note: The function $f(x)$ could have represented the *lineal charge density* along the wire, the amount of charge per unit length. Then "mass" above would be replaced by "charge" so that the total charge Q in the wire is given by

$$Q = \int_a^b f(x)\, dx.$$

2. **Total work done by a nonconstant force:** Suppose that a force $\mathbf{F}(x) = f(x)\mathbf{i}$ is acting on a mass m, resulting in the displacement of the mass from position $x = a$ to $x = b$. We wish to compute the total work done by the force. Proceeding in the same way as above, we form the

partition of the interval $[a, b]$ into n subintervals of length Δx. We then approximate the work ΔW_k done by the force in moving the mass from x_{k-1} to x_k by

$$\Delta W_k \approx f(x_k^*)\Delta x.$$

In other words, we are making the approximation that the force $f(x)$ exerted on the mass over the interval $[x_{k-1}, x_k]$ is constant, i.e., $f(x) \approx f(x_k^*)$. The total work W done by the force is

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k=1}^{n} f(x_k^*)\Delta x = S_n.$$

In the limit $n \to \infty$, the total work become

$$W = \int_a^b f(x)\, dx.$$

It is important to keep these simple examples in mind. They represent the "Spirit of Calculus" which will be used to solve more complicated problems.

As a preview of things to come, suppose that the wire in Example 1 above is no longer straight but has the shape of a curve C in $\mathbf{R}^3$. If we know the lineal mass/charge density function $f(x, y, z)$ along the wire, how can we compute the total mass/charge of the wire?

And for an even more complicated problem, suppose that a nonconstant force $\mathbf{F}(x, y, z)$ is acting on a mass m, causing it to move along a curve $\mathbf{r}(t)$ in $\mathbf{R}^3$. What is the total work done by the force on the mass?

These are motivations for the study of path integrals of scalar and vector-valued functions.

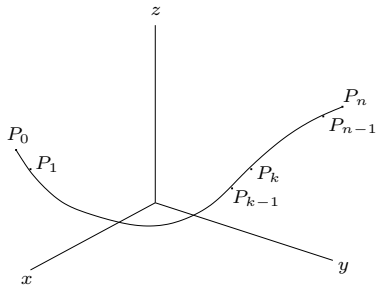
Line integrals of scalar-valued functions

Given a curve C with endpoints P and Q in $\mathbf{R}^3$. Now suppose that there is a scalar valued function $f : \mathbf{R}^3 \to \mathbf{R}$ that is defined at all points on the curve C. For example, the curve C could represent a thin wire and $f(x, y, z)$ could represent the lineal charge density at a point (x, y, z) on the wire.

Let us now introduce points $P_0 = P$, P_1, P_2, ... P_{n-1}, $P_n = Q$ on the curve as shown below.

Let Δs_k denote the length of each "subcurve" C_k with endpoints P_{k-1} and P_k. Now on each subcurve C_k pick a sample point (x_k^*, y_k^*, z_k^*) and compute the product

$$f(x_k^*, y_k^*, z_k^*)\Delta s_k. \tag{3}$$



If f were the lineal charge density, then the above amount would be the approximate amount of charge Δq_k on the subcurve C_k. We have once again made the approximation that the lineal density f is constant over the subcurve.

Now form the sum

$$S_n = \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k. \tag{4}$$

If f is "sufficiently nice," i.e piecewise continuous, and C as well is "sufficiently nice," i.e. piecewise C^1, then the $\lim_{n\to\infty} S_n$ exists, provided that all the $\Delta s_k \to 0$. We denote this limit as

$$\int_C f\ ds \tag{5}$$

and call it the "line integral of f over curve C". If $f = 1$ on C, then

$$\int_C f\ ds = \int_C ds = L, \text{ the length of curve } C. \tag{6}$$

You have seen this computation in Math 128 or equivalent.

The problem of how to practically compute this line integral still remains. The solution lies in finding a parametrization of the curve, i.e.,

$$\mathbf{x}(t) = \mathbf{r}(t) = (x(t), y(t), z(t)), \qquad t \in [a, b].$$

Now divide the interval $[a, b]$ into n subintervals of length $\Delta t = (b - a)/n$ via the partition points $t_k = k\Delta t$, $k = 0, 1, 2, \cdots, n$. We now let the points P_k introduced earlier correspond to the points $\mathbf{r}(t_k) = (x(t_k), y(t_k), z(t_k))$.

From each subinterval $[t_{k-1}, t_k]$, pick a sample point t_k^* so that

$$(x_k^*, y_k^*, z_k^*) = \mathbf{r}(t_k^*) = (x(t_k^*), y(t_k^*), z(t_k^*))$$

is the sample point at which we evaluate $f(x, y, z)$. We now form the sum

$$\begin{aligned} T_n &= \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \\ &= \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*), z(t_k^*))\Delta s_k. \end{aligned} \tag{7}$$

There is still a problem – we have to determine the lengths Δs_k of the subcurves C_k. The first step is to approximate these lengths by the lengths of the straight line segments $\overline{P_{k-1}P_k}$. Then, by Pythagoras,

$$\Delta s_k \approx \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}, \tag{8}$$

where

$$\Delta x_k = x(t_k) - x(t_{k-1}), \quad \Delta y_k = y(t_k) - y(t_{k-1}), \quad \Delta z_k = z(t_k) - z(t_{k-1}). \tag{9}$$

We still need to bring the parameter t into the expression for Δs_k. We do this by multiplying and dividing by Δt:

$$\Delta s_k \approx \sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2 + \left(\frac{\Delta z_k}{\Delta t}\right)^2}\,\Delta t. \tag{10}$$

Now insert this term into (7) to give

$$S_n = \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*), z(t_k^*))\sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2 + \left(\frac{\Delta z_k}{\Delta t}\right)^2}\,\Delta t. \tag{11}$$

This is a Riemann sum. In the limit $n \to \infty$, we have $\Delta t \to 0$ so that the points t_{k-1} and t_k approach each other. The square root term is an approximation to the magnitude of the velocity associated with the parametrization. In the limit $\Delta t \to 0$, it becomes the speed $\|\mathbf{r}(t)\|$. Once again, for f and C "sufficiently nice," the limit of the S_n exists and is the desired line integral:

$$\lim_{n\to\infty} S_n = \int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt. \tag{12}$$

The final integral on the right is an integration over the parameter t. There are a couple of important points regarding this integral:

1. The term $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$ represents the function f being evaluated *over points on the curve C*.

2. The product $\|\mathbf{r}'(t)\|dt = ds$, the infinitesimal element of arclength associated with the parametrization $\mathbf{r}(t)$. It is simply the statement that the infinitesimal displacement along the curve ds is given by the product of the instantaneous speed $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$ with the element of time dt.

Special case: When $f(x,y,z) = 1$, the line integral over curve C reduces to

$$\int_C f\, ds = \int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \|\mathbf{v}(t)\|\, dt = L, \quad \text{the length of curve } C. \tag{13}$$

You should recall this formula from first-year calculus. In fact, most of the derivation of the above line integration is the same as what you saw in first-year calculus – the only modification is the multiplication of the lengths Δs_k of the subcurves by sample values of the function f over those subcurves.

Some examples of line integrals of scalar fields

In what follows we shall compute $\int_C f(x,y)\, ds$ on various curves in $\mathbf{R}^2$ that start at (0,0) and end at (0,1). We shall use the function $f(x,y) = x^2 + y^2$.

1. C is the straight line from (0,0) to (1,1), parametrized as

   $$x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1.$$

   In compact form, the parametrization is $\mathbf{r}(t) = (t,t)$. The velocity is $\mathbf{r}'(t) = (1,1)$ so that the speed is $\|\mathbf{r}'(t)\| = \sqrt{2}$. The element of arclength is given by $ds = \sqrt{2}dt$.

   We must still evaluate the function $f(x,y)$ over the curve:

   $$f(\mathbf{r}(t)) = f(x(t), y(t)) = x(t)^2 + y(t)^2 = t^2 + t^2 = 2t^2.$$

   Therefore the desired line integral becomes

   $$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, ds = \int_0^1 2t^2\sqrt{2}dt = 2\sqrt{2}\int_0^1 t^2 dt = \frac{2\sqrt{2}}{3}. \tag{14}$$

2. Let us now compute the line integral over the straight line from (0,0) to (1,1) but now parametrized as

   $$x(t) = t^2, \quad y(t) = t^2, \quad 0 \le t \le 1.$$

   This means that $\mathbf{r}(t) = (t^2, t^2)$ so that the velocity is $\mathbf{r}'(t) = (2t, 2t)$ and the speed is

   $$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4t^2} = 2\sqrt{2}t.$$

   Thus, $ds = 2\sqrt{2}dt$.

Now evaluate the function $f(x,y)$ over the curve:

$$f(\mathbf{r}(t)) = f(x(t),y(t)) = x(t)^2 + y(t)^2 = t^4 + t^4 = 2t^4.$$

The line integral becomes

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, ds = \int_0^1 2t^4 2\sqrt{2}dt = 4\sqrt{2}\int_0^1 t^5 dt = \frac{2\sqrt{2}}{3}. \tag{15}$$

Note that we arrive at the same result as in 1. above. This is not a conclusive proof but we do state the general result that a line integral is independent of the parametrization used to evaluate the line integral. In retrospect, this is a very natural/desirable result. After all, the parametrization is only a book-keeping method of keeping track of points that lie on the curve and how they are visited. Changing parametrizations can change the velocity, which determines how short or long a time is spent over a particular interval, based on the parametrization. The element of arclength $ds = \|\mathbf{r}'(t)\|dt$ performs the necessary book-keeping.

3. We now consider the following curve from (0,0) to (1,1),

$$x(t) = t, \quad y(t) = t^2, \quad 0 \le t \le 1.$$

This curve lies on the parabola $y = x^2$. Then $\mathbf{r}(t) = (t, t^2)$, implying that $\mathbf{r}'(t) = (1, 2t)$ so that $\mathbf{r}'(t) = \sqrt{1+4t^2}$.

Evaluating the function $f(x,y)$ over the curve:

$$f(\mathbf{r}(t)) = f(x(t),y(t)) = x(t)^2 + y(t)^2 = t^4 + t^4.$$

The line integral becomes:

$$\begin{aligned}\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, ds &= \int_0^1 (t^2+t^4)\sqrt{1+4t^2}\, dt \\ &= \frac{349}{768} - \frac{7}{512}\ln(1+\frac{1}{2}\sqrt{5}) - \frac{7}{512}\ln 2.\end{aligned} \tag{16}$$

This result is different from those of Examples 1 and 2 because of the different path taken.

4. Let us now consider the following example in $\mathbf{R}^3$: The curve $\mathbf{r}(t) = (t, 2t^2, 3t^2)$ for $t \in [0,1]$. The velocity is $\mathbf{r}'(t) = (1, 4t, 6t)$ so that the speed is $\|\mathbf{r}'(t)\| = \sqrt{1+52t^2}$.

Consider the function $f(x,y,z) = x^2 + xyz + z^3$. Then, evaluating f over the curve:

$$f(x(t),y(t),z(t)) = t^2 + 6t^5 + 27t^6.$$

Thus,

$$\begin{aligned}\int_C f\ ds &= \int_0^1 f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt \\ &= \int_0^1 (t^2+6t^5+27t^6)\sqrt{1+52t^2}dt\end{aligned} \tag{17}$$

The approxmate numerical value of this parameter is 32.717 (Maple).

Applications of the line integral

1. The average value of a function $f(x,y,z)$ over a curve C is given by

$$\bar{f}_C = \frac{\int_C f\ ds}{\int_C\ ds} = \frac{\int_C f\ ds}{L}, \tag{18}$$

where L is the length of the curve C. This is a generalization of the definition of the average value of a function $f(x)$ over the interval $[a,b]$.

Going back to Examples 1 and 2 of this section, we computed the line integral of $f(x,y)=x^2+y^2$ over the straight line segment joining $(0,0)$ to $(1,1)$ as $2\sqrt{2}/3$. The length of line segment is $\sqrt{2}$. Therefore the average value of $f(x,y)$ on this line segment is

$$\bar{f}_C = \frac{2\sqrt{2}}{3}\cdot\frac{1}{\sqrt{2}} = \frac{2}{3}. \tag{19}$$

Note: The average value is **not** 1/2, due to the nonlinear nature of the function x^2+y^2.

2. If $\rho(x,y,z)$ is the lineal mass/charge density of a thin wire that has the shape of a curve C in $\mathbf{R}^3$, then the total mass/charge of the wire is given by

$$M = \int_C dm = \int_C \rho\ ds. \tag{20}$$

In the case that ρ represents mass density, then the center of mass, $(\bar{x},\bar{y},\bar{z})$ of the wire is given by by the following formulas,

$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_C x dm = \frac{1}{M}\int_C x\rho\ ds, \\ \bar{y} &= \frac{1}{M}\int_C y dm = \frac{1}{M}\int_C y\rho\ ds, \\ \bar{z} &= \frac{1}{M}\int_C z dm = \frac{1}{M}\int_C z\rho\ ds.\end{aligned} \tag{21}$$

To illustrate, let's return to Example 1 done earlier: The integration of the function $f(x,y)=x^2+y^2$ over the straight line from $(0,0)$ to $(1,1)$. We now consider this straight line to represent

a wire of length $\sqrt{2}$. We'll also consider $\rho(x, y) = f(x, y) = x^2 + y^2$ to be the linear mass density function of the wire. At one end, $\rho(0, 0) = 0$ and at the other end $\rho(1, 1)$ – as one moves from $(0, 0)$ to $(1, 1)$, the density of the wire increases. As such, we would expect that the center of mass would not be at the centroid of the wire, i.e., $(1/2, 1/2)$ but closer to $(1, 1)$ instead.

We have already computed the total mass of the wire in Example 1:

$$M = \int_C dm = \int_C \rho \, ds = \frac{2\sqrt{2}}{3}. \tag{22}$$

It now remains to compute the x and y moments,

$$\int_C x \, dm = \int_C x\rho \, ds. \tag{23}$$

In this case, the integrand evaluated over the curve C will be

$$x(t)\rho(x(t), y(t)) = x(t)[x(t)^2 + y(t)^2] = t(2t^2) = 2t^3. \tag{24}$$

The line integral in (23) becomes

$$\int_C x\rho \, ds = \int_0^1 (2t^3)\sqrt{2} \, dt = 2\sqrt{2} \int_0^1 t^3 \, dt = \frac{\sqrt{2}}{2}. \tag{25}$$

The x-coordinate of the center of mass is then

$$\bar{x} = \frac{\sqrt{2}/2}{2\sqrt{2}/3} = \frac{3}{4}. \tag{26}$$

In a similar fashion, we find that

$$\bar{y} = \bar{x} = \frac{3}{4}. \tag{27}$$

As expected, the center of mass lies closer to the endpoint $(1, 1)$. Moreover, because the wire is straight, the center of mass lies on the wire. (If the wire were bent, this would not be guaranteed.)

Note: It would be an interesting, but possibly tedious, exercise to compute the center of mass of the parabolic wire of Example 3 with mass density function $\rho(x, y) = x^2 + y^2$.

Lecture 28

Line integrals of vector-valued functions

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.2, pp. 1041-1043)

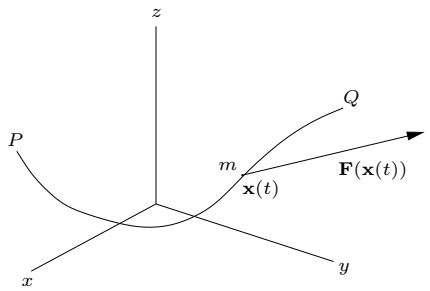
The integration of a vector field $\mathbf{F}$ over a curve C is a commonly encountered application of vector calculus. It can be used to compute the work done by a nonconstant force acting on a particle that moves along a curve. It can also be used to measure the circulation of a rotating fluid or a magnetic field.

In what follows, we shall motivate our treatment with reference to the computation of work done by a force. As such, it will be useful to recall a fundamental formula from physics. Suppose that a *constant* force $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is applied to a mass m that is constrained to move only in the x-direction. The mass is moved from position $x = a$ to $x = b$. Most of you will quickly answer that the total work done by the force is $W = F_1(b - a)$, which is correct. It is the component of the force $\mathbf{F}$ in the direction of motion that "does the work." This is an example of the more general formula that can be used in $\mathbf{R}^n$:

$$W = \mathbf{F} \cdot \mathbf{d}, \tag{28}$$

where $\mathbf{F}$ is the *constant* force acting on an object and $\mathbf{d}$ is the displacement vector $\overrightarrow{PQ}$ of the mass, where P and Q are, respectively, the initial and final positions of the mass. There is another assumption behind this formula: *that the actual motion of the mass was along the straight line* $\overline{PQ}$.

Now let us consider a more general case: A force, $\mathbf{F}(\mathbf{r})$, which is not necessarily constant in space, is acting on a mass m, as the mass moves along a curve C from point P to point Q as shown in the diagram below.



The goal is to compute the total amount of work W done by the force. Clearly Eq. (28) does not apply here. But the fundamental idea, in the "Spirit of Calculus," is to break up the motion into tiny

pieces over which we can use (28) as an approximation. We then "sum up," i.e., integrate, over all contributions to obtain W.

Just as we did for the previous line integration problem, we assume that the curve C can be parametrized, i.e.,

$$\mathbf{x}(t) = \mathbf{r}(t) = (x(t), y(t), z(t)), \qquad t \in [a, b], \tag{29}$$

so that $\mathbf{r}(a)$ is point P and $\mathbf{r}(b)$ is point Q. Now divide the parameter interval $[a, b]$ into n subintervals of length $\Delta t = (b - a)/n$: We do this by defining the partition points $t_k = a + k\Delta t$, $k = 0, 1, 2, \cdots, n$. These points define a set of $n + 1$ points $P_k = \mathbf{r}(t_k) = (x(t_k), y(t_k), z(t_k))$, $k = 0, 1, \cdots, n$ that lie on the curve C, with $P_0 = P$ and $P_n = Q$.

We now come to the major point of this procedure. We shall make two approximations in the computation of the work ΔW_k done by the force $\mathbf{F}$ in moving mass m from point P_{k-1} to point P_k:

1. We shall assume that the force acting on the mass during this time interval is *constant*. From each parameter subinterval $[t_{k-1}, t_k]$, pick a sample point t_k^* so that

   $$(x_k^*, y_k^*, z_k^*) = \mathbf{r}(t_k^*) = (x(t_k^*), y(t_k*), z(t_k^*))$$

   is a sample point on the curve extending from P_{k-1} to P_k Now evaluate the force vector $\mathbf{F}$ at this sample point, i.e., $\mathbf{F}(\mathbf{r}(t_k))$. This is the constant force that we shall assume acts on the mass from P_{k-1} to P_k. (For n very large, we don't expect the force $\mathbf{F}$ to change appreciably over this subcurve.)

2. We shall also assume that the motion of the mass is along the straight line segment $\overline{P_{k-1}P_k}$.

These two approximations now allow us to use the "$\mathbf{F} \cdot \mathbf{d}$" formula in (28): the work ΔW_k done by the force over the subcurve $P_{k-1}P_k$ is approximated by

$$\Delta W_k \approx \mathbf{F}(\mathbf{r}(t_k^*)) \cdot \overrightarrow{P_{k-1}P_k}. \tag{30}$$

We still need to express the displacement vector $\overrightarrow{P_{k-1}P_k}$ in terms of the parameter t, in order to be able to produce an integration over t. Note that

$$\begin{aligned} \overrightarrow{P_{k-1}P_k} &= \mathbf{r}(t_k) - \mathbf{r}(t_{k-1}) \\ &= \frac{\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})}{\Delta t}\Delta t \\ &\approx \mathbf{r}'(t_k^*)\Delta t. \end{aligned} \tag{31}$$

The final line is another approximation – we have replaced the *mean velocity* over the line segment $\overrightarrow{P_{k-1}P_k}$ with the value of the velocity at the sample point $\mathbf{r}(t_k^*)$. However, as n is increased, the length of the interval $[t_{k-1}, t_k]$ decreases, and the approximation gets better.

Thus the work ΔW_k is now approximated by

$$\Delta W_k \approx \mathbf{F}(\mathbf{r}(t_k^*)) \cdot \mathbf{r}'(t_k^*)\Delta t. \tag{32}$$

The total work is now approximated as a sum over all subcurves:

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k-1}^{n} \mathbf{F}(\mathbf{r}(t_k^*)) \cdot \mathbf{r}'(t_k^*)\Delta t. \tag{33}$$

This is now a Riemann sum involving the *scalar-valued function*

$$f(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \tag{34}$$

evaluated at the sample points t_k^*. In the limit $n \to \infty$, this Riemann sum converges to a definite integral:

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \text{“ } \int_C \mathbf{F} \cdot \ d\mathbf{r}. \text{ ”}. \tag{35}$$

The term

$$d\mathbf{r} = \mathbf{r}'(t)dt \tag{36}$$

represents the *infinitesimal displacement vector* associated with the velocity vector $\mathbf{r}'(t)$. The expression “$\int_C \mathbf{F} \cdot d\mathbf{r}$” denotes the “line integral of the vector field $\mathbf{F}$ over the curve C. But it will be the expression,

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt, \tag{37}$$

that will be useful for the practical computation of these line integrals.

A comment regarding notation:

Various notations are implied for the path integral of a vector-valued function. You may see the following in other books:

$$\int_C \mathbf{F} \cdot d\mathbf{x}, \qquad \int_C \mathbf{F} \cdot d\mathbf{s}. \tag{38}$$

They all mean the same thing – the line integral of the vector field $\mathbf{F}$ over the curve C.

Another interesting note which was not covered in the lecture:

Before going on to some examples, let us perform a slight modification of the above expression for the line integral of a vector field. We shall simply multiply and divide the integrand by the speed $\| \mathbf{r}(t) \|$ associated with the parametrization:

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} \| \mathbf{r}'(t) \| \, dt \tag{39}$$

But

$$\frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} = \hat{\mathbf{T}}(t), \tag{40}$$

The *unit tangent vector* to the curve C at $\mathbf{r}(t)$. And

$$\| \mathbf{r}'(t) \| \, dt = ds, \tag{41}$$

the infinitesimal element of arclength along the curve C. The result is that

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \hat{\mathbf{T}}(t) \| \mathbf{r}'(t) \| \, dt = \int_C \mathbf{F} \cdot \hat{\mathbf{T}} \, ds. \tag{42}$$

The scalar quantity $\mathbf{F} \cdot \hat{\mathbf{T}}$ is the projection of $\mathbf{F}$ in the direction of $\hat{\mathbf{T}}$, which is precisely the component of $\mathbf{F}$ doing the work in moving the mass in the direction of $\hat{\mathbf{T}}$, which is the direction of the velocity vector $\mathbf{r}'(t)$, the instantaneous direction of motion of the mass at any point on the curve.

In a more general setting, the above line integral is adding up the component of $\mathbf{F}$ in the direction of the tangent vector along the curve. We shall return to this important concept later in this chapter.

Some examples:

1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ along the curve $\mathbf{r}(t) = (t, t, t)$ for $0 \leq t \leq 1$.

   This parametrization produces a straight line that starts at (0,0,0) and ends at (1,1,1).

   Step 1: Evaluate the velocity vector: $\mathbf{r}'(t) = (1, 1, 1)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Dropping the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for convenience, we simply write $\mathbf{F} = (xyz, y^2, z)$ so that

   $$\mathbf{F}(\mathbf{r}(t)) = (x(t)y(t)z(t), y(t)^2, z(t)) = (t^3, t^2, t). \tag{43}$$

Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (t^3, t^2, t) \cdot (1, 1, 1) = t^3 + t^2 + t. \tag{44}$$

We may now evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^3 + t^2 + t)dt = \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{13}{12}. \tag{45}$$

2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ as in Example 1, but the curve is now $\mathbf{r}(t) = (t, t^2, t^2)$, $0 \leq t \leq 1$

   This parametrization produces a parabolic curve line that also starts at (0,0,0) and ends at (1,1,1).

   Step 1: Evaluate the velocity vector: $\mathbf{r}'(t) = (1, 2t, 2t)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Here,

$$\mathbf{F}(\mathbf{r}(t)) = (x(t)y(t)z(t), y(t)^2, z(t)) = (t^4, t^2, t^2). \tag{46}$$

   Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (t^5, t^4, t^2) \cdot (1, 2t, 2t) = t^5 + 2t^5 + 2t^3 = 3t^5 + 2t^3. \tag{47}$$

   We may now evaluate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^5 + 2t^3)dt = \frac{1}{2} + \frac{1}{2} = 1. \tag{48}$$

   Note that this result differs from that of Example 1. The line integrals had the same endpoints but different paths. There is no guarantee that the results will be the same.

Lecture 29

Line integrals of vector-valued functions (cont'd)

An important application in Physics

Proposition: A mass m moves in $\mathbf{R}^3$ under the influence of a force $\mathbf{F}$ according to Newton's Law $\mathbf{F} = m\mathbf{a}$. The mass moves from A to point B along a trajectory $\mathbf{r}(t)$ which we shall denote as curve C_{AB}. Then the work W done by the force $\mathbf{F}$ along C_{AB} is equal to the change in kinetic energy of the mass, i.e.

$$W = K(B) - K(A) = \Delta K, \tag{49}$$

where K denotes the kinetic energy of the particle.

Proof: The total work done by the force is

$$\begin{aligned} W &= \int_{C_{AB}} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt, \end{aligned} \tag{50}$$

where a and b denote the times that the particle is at A and B, respectively. But $\mathbf{r}'(t) = \mathbf{v}(t)$, the velocity of the mass. At all points on the trajectory, Newton's Law is obeyed, implying that

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{a}(t) = m\mathbf{v}'(t). \tag{51}$$

We substitute this result into the work integral:

$$W = \int_a^b m\mathbf{v}'(t) \cdot \mathbf{v}(t)\, dt. \tag{52}$$

But recall that

$$\frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) + \mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}'(t) \cdot \mathbf{v}(t). \tag{53}$$

or

$$\mathbf{v}'(t) \cdot \mathbf{v}(t) = \frac{1}{2}\frac{d}{dt}\mathbf{v}(t) \cdot \mathbf{v}(t) = \frac{1}{2}\frac{d}{dt} \parallel \mathbf{v}(t) \parallel^2 . \tag{54}$$

Therefore

$$\begin{aligned} W &= \frac{1}{2}m\int_a^b \frac{d}{dt} \parallel \mathbf{v}(t) \parallel^2 \; dt \\ &= \frac{1}{2}m \parallel \mathbf{v}(b) \parallel^2 - \frac{1}{2}m \parallel \mathbf{v}(a) \parallel^2 \\ &= K(B) - K(A) \\ &= \Delta K. \end{aligned} \tag{55}$$

You probably saw this result for one-dimensional motion in your first-year Physics course. Note, however, that we have **not** made any assumptions on $\mathbf{F}$ – it does not have to be conservative. For example, it also holds for frictional forces, which are nonconservative. The proof of this result for *conservative* forces is much simpler, since we can use the fact that total mechanical energy is conserved. But we must use the Generalized Fundamental Theorem of Calculus, the next topic.

The Generalized Fundamental Theorem of Calculus

(Relevant section from Stewart, *Calculus, Early Transcendentals*, Sixth Edition: 16.3)
We begin with some additional examples of line integrals of vector-valued functions, which will motivate the discussion.

Examples:

1. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = 2x\mathbf{i}+4y\mathbf{j}+z\mathbf{k}$ along the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$, with $0 \leq t \leq 2\pi$.

   This parametrization produces a helical curve that starts at $(1,0,0)$ and ends at $(1,0,2\pi)$.

   Step 1: Evaluate the velocity vector: $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve:

   $$\mathbf{F}(\mathbf{r}(t)) = (2x(t), 4y(t), z(t)) = (2\cos t, 4\sin t, t) \tag{56}$$

   Step 3: Now construct the dot product that will appear in the integrand:

   $$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (2\cos t, 4\sin t, t) \cdot (-\sin t, \cos t, 1) = 2\cos t \sin t + t \tag{57}$$

Now evaluate the line integral:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (2\cos t \sin t + t)\, dt \\ &= \left[\sin^2 t + \frac{1}{2}t^2\right]_0^{2\pi} \\ &= 2\pi^2.\end{aligned} \tag{58}$$

2. Now evalute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field used in Example 1, but the curve is now the straight line from $(1,0,0)$ to $(1,0,2\pi)$. Since the value of the line integral is independent of the parametrization used, we'll use the simplest one, $\mathbf{r}(t) = (1,0,t)$, with $0 \le t \le 2\pi$.

   Step 1: Evaluate the velocity vector: $\mathbf{r}'(t) = (0,0,1)$.

   Step 2: Evaluate $\mathbf{F}$ at points on the curve, using the parametrization. Here,

$$\mathbf{F}(\mathbf{r}(t)) = (2x(t), 4y(t), z(t)) = (2,0,t). \tag{59}$$

   Step 3: Now construct the dot product that will appear in the integrand:

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = (2,0,t)\cdot(0,0,1) = t \tag{60}$$

   Now evaluate the line integral:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} t\, dt = 2\pi^2. \tag{61}$$

Note that the results of Examples 1 and 2 are identical. This could be a coincidence but if you tried other paths with the same endpoints, you will obtain $2\pi^2$. We'll show very shortly that for vector field $\mathbf{F} = (x, 2y, 4z)$, the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi^2 \tag{62}$$

for *any* (piecewise C^1) path that starts at $(1,0,0)$ and ends at $(1,0,2\pi)$. In other words, *the line integral is independent of path.*

The reason for this independence of the path is the fact that the vector field $\mathbf{F}$ is *conservative*, i.e. there exists a scalar function $V(x,y,z)$ such that $\mathbf{F} = -\vec{\nabla}V$. Recall that mathematicians prefer to say the the vector field is a *gradient field*, i.e., there exists a scalar function $f(x,y,z)$ such that

$\mathbf{F} = \vec{\nabla} f$. In the discussion that follows, we shall use the idea of gradient fields, but always with the understanding that they are equivalent to conservative fields, with the relation

$$f(x, y, z) = -V(x, y, z). \tag{63}$$

Now returning to the vector field $\mathbf{F}$ used above, note that

$$\mathbf{F} = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k} = \vec{\nabla} f, \tag{64}$$

where

$$f(x, y, z) = x^2 + 2y^2 + \frac{1}{2}z^2. \tag{65}$$

Here is our claim:

Let C be a (piecewise C^1) curve $\mathbf{r}(t)$, $a \leq t \leq b$, and $\mathbf{F} = \vec{\nabla} f$ a gradient vector field over a region $D \subset \mathbf{R}^n$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \vec{\nabla} f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{66}$$

This result is known as the *Generalized Fundamental Theorem of Calculus* (GFTC) for functions of several variables. *Note that the line integral depends only on the endpoints and not on the path taken.* We'll often state this result as follows:

$$\int_{C_{AB}} \mathbf{F} \cdot d\mathbf{r} = \int_C \vec{\nabla} f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \text{“}f(B) - f(A)\text{”}, \tag{67}$$

where C_{AB} denotes a curve that starts at A and ends at B, and $f(A)$ and $f(B)$ denote the values of f at these respective endpoints.

Note: The above result is almost universally known as the “Generalized Fundamental Theorem of Calculus.” In Stewart's textbook, Section 16.3, it is referred to as “The Fundamental Theorem for Line Integrals.”

Why “Generalized Fundamental Theorem”? Let's go back to the Fundamental Theorem of Calculus for functions of a single variable, which implies that

$$\int_a^b f'(x)\, dx = f(b) - f(a), \tag{68}$$

since $f(x)$ is an antiderivative of $f'(x)$. Comparing (68) and (66) it appears that the gradient of f, $\vec{\nabla} f$, is the "natural derivative" of a function f of several variables. (Of course, for the single variable case, it is the derivative: $\vec{\nabla} f = f'(x)\mathbf{i}$.)

We now prove the GFTC in (66) for the particular case $n = 3$:

$$\begin{aligned}
\int_C \vec{\nabla} f \cdot d\mathbf{r} &= \int_a^b \vec{\nabla} f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).
\end{aligned} \tag{69}$$

Revisiting Examples 1 and 2 above: Let us now return to the vector field $\mathbf{F}$ used in Examples 1 and 2, and the knowledge that it is a gradient field, as shown in Eq. (65). From the GFTC, the line integral will simply be the difference of the function f evaluated at the endpoints:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \vec{\nabla} f \cdot d\mathbf{r} \\
&= \left[x^2 + 2y^2 + \frac{1}{2} z^2 \right]_{(1,0,0)}^{(1,0,2\pi)} \\
&= 2\pi^2.
\end{aligned} \tag{70}$$

As such, the value of the line integral will not depend on the path that is taken from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

How do we know if a vector field is gradient/conservative?

Obviously, the GFTC gives a nice way of computing line integrals involving vector fields, but only if the vector fields are conservative. In fact, the consequence of independence of path is of fundamental importance to physics (next lecture), even if we are not so concerned about explicitly computing actual values of line integrals.

There naturally arise two questions:

1. If one is presented with a line integral involving a vector field $\mathbf{F}$, how does one know if $\mathbf{F}$ is a gradient/conservative field?

2. If $\mathbf{F}$ is indeed a gradient/conservative field, how do we find f/V?

Actually, we answered these questions in an earlier lecture:

Answer to 1: The vector field is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$ if

$$\vec{\nabla} \times \mathbf{F} = \mathbf{0}. \tag{71}$$

Answer to 2: We obtain f or V from $\mathbf{F}$ by means of a systematic procedure of antidifferentiation. So far, we've considered only the two-dimensional case, i.e., fields in $\mathbf{R}^2$.

Another important consequence of conservative fields in Physics

Recall that if a vector field $\mathbf{F}$ is a gradient field, i.e., $\mathbf{F} = \vec{\nabla} f$, then it is also a conservative field, $\mathbf{F} = -\vec{\nabla} V$, with $V = -f$. (Actually, $V = f + C$, where C is a constant.) Suppose that a conservative force acts upon a mass m while it is moving from point A to point B along a curve C_{AB} in $\mathbf{R}^3$. Then the total work done by the force is given by

$$\begin{aligned} W &= \int_{C_{AB}} \mathbf{F} \cdot d\mathbf{r} \\ &= -\int_{C_{AB}} \vec{\nabla} V \cdot d\mathbf{r} \qquad (\mathbf{F} = -\vec{\nabla} V) \\ &= -[V(B) - V(A)] \qquad (\text{by GFTC}) \\ &= V(A) - V(B) \\ &= -\Delta V. \end{aligned} \tag{72}$$

Here, $V(A)$ and $V(B)$ denote the potential energies of the mass at A and B, respectively, and ΔV denotes the change in the potential energy from A to B. Note that we did not have to know the curve C_{AB} along which the mass travelled. The above result would have been valid for *any* curve that started at A and ended at B.

Now since $\mathbf{F}$ was assumed to be conservative, the total mechanical energy of the mass is conserved, i.e.,

$$K(A) + V(A) = K(B) + V(B), \tag{73}$$

where $K(A)$ and $K(B)$ denote the kinetic energies of the mass at points A and B, respectively. Rearranging this equation, we have

$$V(A) - V(B) = K(B) - K(A), \tag{74}$$

so that, from (72),

$$W = K(B) - K(A). \tag{75}$$

The total work done by $\mathbf{F}$ is equal to the change in kinetic energy of the mass. This result was derived *for general forces, conservative and nonconservative alike,* in a previous lecture.

In summary, we have the following result for the total work done by the force $\mathbf{F}$ when the mass m moves from A to B:

$$W = \Delta K = -\Delta V. \tag{76}$$

This makes sense: the total mechanical energy E of the mass remains constant. Any increase/decrease in its kinetic energy K must be accompanied by a decrease/increase in its potential energy V.

We have already encountered a number of examples of physical forces that are conservative. For example, in an earlier lecture we discussed a two-dimensional mass-spring system, where the force exerted on a mass m was given by

$$\mathbf{F}(x, y) = -k_1 x\mathbf{i} - k_2 y\mathbf{j}, \tag{77}$$

where (x, y) denotes the position of the mass relative to the equilibrium point. This force is a conservative, and its associated potential energy function is

$$V(x, y) = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + C, \tag{78}$$

where C is a constant. Suppose that the mass were observed to be at position A with coordinates (a, b) at one time and at B (c, d) at another time. Then the net work done by the force in the net movement of the mass from A to B is, from Eq. (72),

$$W = V(A) - V(B) = \frac{1}{2}k_1(a^2 - c^2) + \frac{1}{2}k_2(b^2 - d^2). \tag{79}$$

If $W > 0$, then this amount is actually the work done *by* the force. If $W < 0$, then $|W|$ is the amount of work that has to be done *against* the force to move the mass from A to B.

As you well know by now, another class of forces that are conservative are those in $\mathbf{R}^3$ that have the form

$$\mathbf{F}(\mathbf{r}) = \frac{K}{r^3}\mathbf{r}, \tag{80}$$

where $r = \| \mathbf{r} \|$. Recalling that

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r}, \tag{81}$$

we have that $\mathbf{F} = -\vec{\nabla}V$, where

$$V(\mathbf{r}) = \frac{K}{r^3}\mathbf{r}. \tag{82}$$

1. $\mathbf{F}(\mathbf{r}) = \dfrac{Qq}{4\pi\epsilon_0 r^3}\mathbf{r}$, the electrostatic force on a charge q at $\mathbf{r}$ due to a charge Q at the origin $\mathbf{0}$. Here, $K = \dfrac{Qq}{4\pi\epsilon_0}$ so that the potential energy function is $V(\mathbf{r}) = \dfrac{Qq}{4\pi\epsilon_0 r}$.
2. $\mathbf{F}(\mathbf{r}) = -\dfrac{GMm}{r^3}\mathbf{r}$, the gravitational force on a mass m at $\mathbf{r}$ due to a mass M at the origin $\mathbf{0}$. Here, $K = -GMm$ so that the potential energy function is $V(\mathbf{r}) = -\dfrac{GMm}{r}$.

It's worth pointing out that earlier in this course, we "discovered" that these forces were conservative from the gradient relation in (81). We were spared the work of trying to find the potential functions associated with these forces, i.e., by first checking if the forces were conservative (using the curl test) and then integrating backwards to find the potential functions.

Here's an application that you've no doubt seen in first-year Physics: Suppose that a satellite moves from point A in space to point B along a curve C_{AB} under the influence of the earth's gravity. What is the work done by gravity?

We assume (and correctly so, as we'll prove later) that we can treat the earth as a point mass M that defines the origin O of our fixed coordinate system. Using the results from a couple of paragraphs above, the answer is simply

$$\begin{aligned} W = V(A) - V(B) &= -\frac{GMm}{r_A} + \frac{GMm}{r_B} \\ &= GMm\left[-\frac{1}{r_A} + \frac{1}{r_B}\right], \end{aligned} \tag{83}$$

where $r_A = |\overline{OA}|$ and $r_B = |\overline{OB}|$ are the radial distances between the earth (origin O) and the satellite at points A and B.

Some notes:

1. If $r_A = r_B$, then $W = 0$.
2. If $r_A > r_B$ (i.e., the satellite has moved inward), then $W > 0$. The work done by gravity is positive (decrease in potential energy).

3. If $r_A < r_B$ (i.e., the satellite has moved outward), then $W < 0$. The work done by gravity is negative (increase in potential energy).

It is certainly possible that all of these situations can be encountered *during a single orbit* of the satellite around the earth. For example,

1. If the orbit is perfectly circular, then $r_A = r_B = r$, a constant, during the entire orbit, which means that no work is ever done.

2. If the orbit is elliptical, then pick two points on the orbit such that $r_A > r_B$. In travelling from A to B, work has been done *by* gravity (decrease in potential energy). In returning to A from B, an equal amount of work has been done *against* gravity (implying an equal increase in potential energy). The motion along an elliptical orbit involves a constant interchange between potential and kinetic energy.