Lecture 21

Sequences (cont'd)

The Monotonic Sequence Theorem

`Relevant section from Stewart: 11.1`

Let us recall a few basic properties of sequences established in the the previous lecture. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be:

- **bounded from above** or simply **bounded above**, if there exists a real number $M \in \mathbb{R}$ such that
$$a_n \leq M \quad \text{for all} \quad n \geq 1\,, \tag{1}$$
- **bounded from below** or simply **bounded below**, if there exists a real number $m \in \mathbb{R}$ such that
$$a_n \geq m \quad \text{for all} \quad n \geq 1\,, \tag{2}$$
- **bounded** if it is bounded both above and below, i.e., there exist real numbers $m, M \in \mathbb{R}$ such that
$$m \leq a_n \leq M \quad \text{for all} \quad n \geq 1\,. \tag{3}$$

Furthermore, a sequence $\{a_n\}_{n=1}^{\infty}$ is said to be:

- **increasing** if
$$a_n < a_{n+1} \quad \text{for all} \quad n \geq 1\,, \tag{4}$$
- **decreasing** if
$$a_n > a_{n+1} \quad \text{for all} \quad n \geq 1\,. \tag{5}$$
- **monotone** if it is either increasing or decreasing.

We might expect that if a sequence $\{a_n\}$ is both **increasing** and **bounded above**, then there is a good chance that it will have a limit L. Its elements are bounded above by some number, i.e., $a_n \leq M$. Furthermore, since the sequence is increasing, i.e., $a_n < a_{n+1}$, the elements of the sequence can't oscillate.

Similarily, we might expect that if a sequence $\{a_n\}$ is both **decreasing** and **bounded below**, then there is a good chance that it would have a limit L.

It turns out that definite statements can be made about such **monotone sequences**. They are included as special cases in the following important theorem:

> **Bounded Monotone Sequence Theorem:** A bounded, monotone sequence is convergent.

We'll prove this theorem. But first we must provide some additional background about the real number system $\mathbb{R}$.

- First of all, a finite set of real numbers, $\{a_n\}_{n=1}^{N} = \{a_1, a_2, \cdots, a_N\}$ always has a **maximum value** and a **minimum value**, which implies that the set is **bounded**, i.e.,

$$m \leq a_n \leq M\,, \quad 1 \leq n \leq N\,, \tag{6}$$

  where

$$m = \min_{1\leq n\leq N} a_n\,, \qquad M = \max_{1\leq n\leq N} a_n\,. \tag{7}$$

- An infinite sequence $\{a_n\}_{n=1}^{\infty}$ may be bounded, but it doesn't have to have a maximum or minimum value. For example, consider the sequence defined as follows,

$$a_n = 1 - \frac{1}{n} = \frac{n-1}{n}\,, \quad n \geq 1\,, \tag{8}$$

  i.e., the set

$$S = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}. \tag{9}$$

  This sequence appears to be increasing. In fact, we can prove this quite easily. For $n \geq 1$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{n}{n+1} - \frac{n-1}{n} \\ &= \frac{n^2 - (n^2-1)}{n(n+1)} \\ &= \frac{1}{n(n+1)} \\ &> 0\,, \end{aligned} \tag{10}$$

which implies that

$$a_{n+1} > a_n \quad n \geq 1 \,. \tag{11}$$

This sequence has a minimum value $a_1 = 0$. But it does **not** have a maximum value. Note, however, that

$$a_n = 1 - \frac{1}{n} = \frac{n-1}{n} < 1 \quad n \geq 1 \,, \tag{12}$$

which implies that the sequence is bounded above – in fact, it is bounded above by $M = 1$. But there is no element a_n in the sequence that equals 1.

In fact, we can easily see that

$$a_n < M \tag{13}$$

for any number $M \geq 1$, e.g., $M = 5$. But $M = 1$ seems to be the smallest such **upper bound**. In fact, $M = 1$ is the so-called **least upper bound** of the sequence since it satisfies the following properties:

- For any $c > M = 1$, $a_n < c$ for all $n \geq 1$,
- For any $c < M = 1$, there exists an a_n such that $a_n > c$.

The least upper bound $M = 1$ may be viewed as the "best" upper bound of the sequence, i.e., the one that is "closest" to the sequence, essentially "touching" it but not necessarily intersecting it.

- The above is a particular case of a more general property of real numbers - referred to by Stewart (Section 11.1, p. 702) as the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers:

  > Let $S \subset \mathbb{R}$ be a nonempty subset of the real numbers that is bounded above, i.e., there exists an $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$, then S has a least upper bound b. This means that if $c < b$, then there exists at least one element $x \in S$ such that $x > c$ (which implies that c is not an upper bound).

  Example: Consider the subset $S = [0, 1)$. But it is bounded above since $x < 1$ for all $x \in S$. But it does not have a maximum value. The least upper bound is $b = 1$ which is not an element of the set.

  Example: Now consider the subset $S = [0, 1]$. It is bounded above since $x \leq 1$ for all $x \in S$. It does have a maximum value, namely, the element $x = 1$. This maximum value is also the least upper bound.

The Completeness Axiom essentially states that there are no "gaps" or "holes" in the real number line. If an infinite set does not have a maximum value, at least it has a least upper bound b which is "arbitrarily close" to the sequence - we can't find a gap between the sequence and the bound b.

The Completeness Axiom also applies to sets that are bounded below:

> Let $S \subset \mathbb{R}$ be a nonempty subset of the real numbers that is bounded below, i.e., there exists an $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in S$, then S has a **greatest lower bound** g. This means that if $c > g$, then there exists at least one element $x \in S$ such that $x < c$ (which implies that c is not a lower bound).

We now prove (one part) of the Monotonic Sequence Theorem.

Proof: First of all, recall that one of the assumptions of the sequence $\{a_n\}$ is that it is **monotone**. Therefore, it is either **increasing** or **decreasing**. We'll consider the case of an increasing sequence. The other assumption on $\{a_n\}$ is that it is **bounded**. From the Completeness Axiom, the set $S = \{a_n \mid n \geq 1\}$ is bounded above and therefore has a least upper bound b. This implies that

$$a_n \leq b \quad \text{for all} \quad n \geq 1\,. \tag{14}$$

But it also implies that for any $\epsilon > 0$ (with the idea of letting ϵ become arbitarily close to zero), $b - \epsilon$ is not an upper bound for the set S, i.e., there exists an $N \geq 1$ such that

$$b - \epsilon < a_n \leq b\,. \tag{15}$$

But recall that the sequence $\{a_n\}$ was assumed to be increasing, i.e., $a_n < a_{n+1}$, which implies that

$$b - \epsilon < a_n \leq b \quad \text{for all} \quad n > N\,. \tag{16}$$

Now subtract b from all terms,

$$-\epsilon < a_n - b \leq 0 \quad \text{for all} \quad n > N\,. \tag{17}$$

Since $0 \leq \epsilon$, we can conclude that

$$-\epsilon < a_n - b \leq \epsilon \quad \text{for all} \quad n > N\,. \tag{18}$$

But this statement is equivalent to the following statement,

$$|a_n - b| < \epsilon \quad \text{for all} \quad n > N\,. \tag{19}$$

Recalling that this holds true for any $\epsilon > 0$, the above statement implies that

$$\lim_{n\to\infty} a_n = b\,. \tag{20}$$

This, of course, implies that the sequence $\{a_n\}$ is convergent.

In the case that the sequence $\{a_n\}$ is **decreasing**, the proof is very similar in style to the above, but using the greater lower bound property of the set S. We'll leave this as an exercise for the reader.

Example: The sequence defined by $a_1 = 0$ and

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{2}\,, \quad n \geq 0\,. \tag{21}$$

We note that

$$a_1 = 0\,, \quad a_2 = \frac{1}{2}\,, \quad a_3 = \frac{3}{4}\,, \quad a_4 = \frac{7}{8}\,. \tag{22}$$

It looks like the sequence $\{a_n\}$ is increasing. In this particular case, it also looks as we might be able to obtain an expression for the a_n in closed form, but we won't bother. We'll simply proceed the more general route and be satisfied with proving that the a_n are increasing. There may be several ways to prove this. Here, we'll prove it by induction.

Clearly, the increasing property $a_n < a_{n+1}$ holds for $n = 1$. Let us now assume that it holds for $n = 1, 2, \cdots N$, i.e.,

$$a_1 < a_2 < \cdots < a_N < a_{N+1}\,. \tag{23}$$

We must show that it holds for $n = N + 1$, i.e.,

$$a_{N+1} < a_{N+2}\,. \tag{24}$$

In the case $n = N + 2$, from the defining recursion relation,

$$a_{N+2} = \frac{1}{2}a_{N+1} + \frac{1}{2}\,. \tag{25}$$

But $a_{N+1} > a_N$ by assumption, so we have that

$$\begin{aligned} a_{N+2} &> \frac{1}{2}a_N + \frac{1}{2} \\ &= a_{N+1}\,. \end{aligned} \tag{26}$$

Since the result holds for $n = N+1$, it holds for all $n \geq 1$. Therefore the sequence is increasing.

We must now show that this increasing sequence is bounded above. Let's see if we can show that

$$a_n < 10 \quad \text{for} \quad n \geq 1\,. \tag{27}$$

Yes, we could probably obtain a "better" upper bound, e.g., $a_n < 2$, but it doesn't really matter. The key is to show that the sequence is bounded above. Once again, we'll try to prove this by induction. Clearly, $a_1 = 0 < 10$, and even $a_2 = \frac{1}{2} < 10$. Let us assume that

$$a_n < 10 \quad \text{for} \;\; n = 1, 2, \cdots, N\,, \tag{28}$$

and try to show that

$$a_{N+1} < 10\,. \tag{29}$$

Once again, from the recursion relation, for $n = N$,

$$a_{N+1} = \frac{1}{2}a_N + \frac{1}{2} < \frac{1}{2} \cdot 10 + \frac{1}{2}\,, \tag{30}$$

where the final line comes from the fact that $a_N < 10$ (by assumption). We therefore conclude that

$$a_{N+1} < \frac{11}{2} < 10\,. \tag{31}$$

The desired boundedness property therefore holds for all $n \geq 1$.

In summary, we have established that the sequence $\{a_n\}$ in this example is (i) increasing and (ii) bounded above. Therefore, from the Bounded Monotone Sequence Theorem, we may conclude that it is a convergent sequence, i.e.,

$$\lim_{n\to\infty} a_n = L \tag{32}$$

exists and is finite. We may find this limit by using the original recursion relation in (21) and taking limits of both sides,

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \left[\frac{1}{2}a_n + \frac{1}{2}\right] \\ &= \frac{1}{2}\lim_{n\to\infty} a_n + \lim_{n\to\infty}\frac{1}{2} \quad \text{(this is valid since the limits exist)}\,. \end{aligned} \tag{33}$$

This implies that

$$L = \frac{1}{2}L + \frac{1}{2}\,, \qquad (34)$$

which is easily solved:

$$L = 1\,. \qquad (35)$$

Note: It is often possible to employ other "tricks" to establish properties such as increasing, decreasing or bounded. For example, to show the increasing property, we may once again start with the induction procedure, i.e., Eq. (23). Instead of going with Eq. (25), however, we simply investigate the difference,

$$\begin{aligned} a_{N+2} - a_{N+1} &= \left[\frac{1}{2}a_{N+1} + \frac{1}{2}\right] - \left[\frac{1}{2}a_N + \frac{1}{2}\right] \\ &= \frac{1}{2}[a_{N+1} - a_N] \\ &> 0\,, \end{aligned} \qquad (36)$$

by assumption in the induction process. The result therefore holds for all $n \geq 1$.

A final note regarding sequences generated by the iteration procedure $x_{n+1} = f(x_n)$

In the past two lectures, we have examined a number of sequences which are obtained by **recursion** or **iteration**. The iteration sequence studied in the previous section is an example:

$$a_1 = 0\,, \qquad a_{n+1} = \frac{1}{2}a_n + \frac{1}{2}\,. \qquad (37)$$

In general, such procedures may be represented by the iteration procedure,

$$x_{n+1} = f(x_n)\,, \qquad n \geq 1\,, \qquad (38)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a specified function. Once a **seed point** a_1 is specified, a unique sequence $\{x_n\}_{n=1}^{\infty}$ will result. For the sequence in (37),

$$f(x) = \frac{1}{2}x + \frac{1}{2}\,. \qquad (39)$$

A variety of behaviours can be exhibited by iteration sequences of functions, ranging from simple convergence to a limit point, convergence to "two-cycles," and n-cycles in general, as well as "chaotic behaviour". These behaviours are certainly of interest mathematically but they are also of interest, and quite useful, in applications to the sciences and engineering.

The mathematical study of iteration sequences and their behaviour can be classified under the general title of **dynamical systems theory**. But differential equations (DEs) can also be classified under this title. The difference between DEs and iteration processes is that the former (DEs) represent a **continuous evolution**, e.g., the position $x(t)$ of a point particle under the action of a force, whereas the latter (iteration) may be viewed as a **discrete evolution** which takes place at discrete time steps $t = n$, $n = 1, 2, \cdots$.

Note: The material presented in the next two sections is optional and only for purposes of information. You are NOT responsible for any of this material for the final exam in this course.

APPENDIX: The Newton-Raphson method as a dynamical system

In 1A Calculus, you most probably studied the **Newton** or **Newton-Raphson** method, a quite famous iteration scheme which was originally designed to provide estimates of (simple) zeros of functions. In what follows, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function, i.e., $f'(x)$ is a continuous function of x on some domain. Now assume that f has a zero at $x = \bar{x}$, i.e., $f(\bar{x}) = 0$. Furthermore assume that this zero is **simple**, i.e., $f'(\bar{x}) \neq 0$. The Newton function associated with f is defined as

$$N(x) = x - \frac{f(x)}{f'(x)} \,. \tag{40}$$

Note that

$$f(\bar{x}) = 0 \quad \Longrightarrow \quad N(\bar{x}) = \bar{x} \,. \tag{41}$$

The zero, $\bar{x}$, is a **fixed point** of the Newton function $N(x)$. It is well known that if we start with a seed point x_0 sufficiently close to $\bar{x}$, then the iteration sequence defined by

$$x_{n+1} = N(x_n) \tag{42}$$

converges to $\bar{x}$, i.e.,

$$\lim_{n\to\infty} x_n = \bar{x} \,. \tag{43}$$

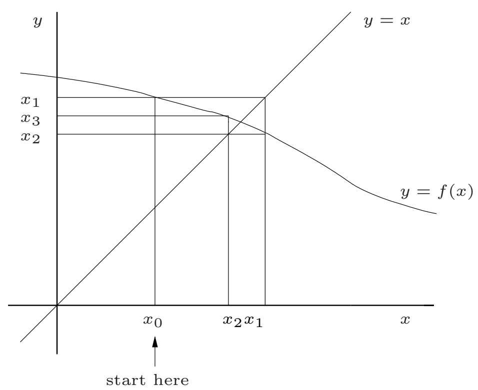
There are some other interesting properties of the iterates in terms of how their distances to $\bar{x}$ go to zero, but that will be the subject of a future discussion.

APPENDIX: Graphical methods of analyzing the iteration scheme $x_{n+1} = f(x_n)$

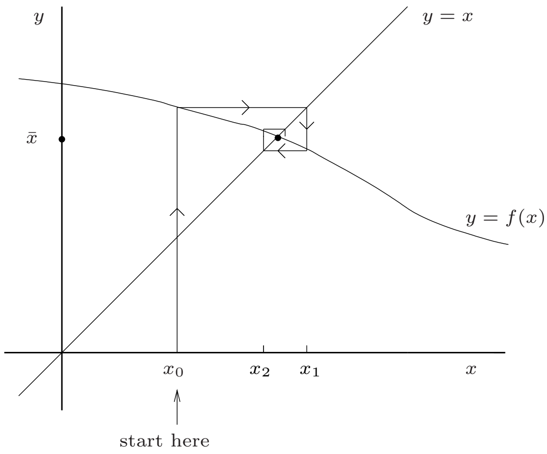
We wish to represent the iteration process defined by

$$x_{n+1} = f(x_n), \tag{44}$$

where $f : \mathbb{R} \to \mathbb{R}$, graphically. We begin with a "seed" $x_0 \in \mathbb{R}$ and insert it into the "f machine". The output $x_1 = f(x_0)$ is then reinserted into the "f machine" to produce $x_2 = f(x_1)$, etc. How do we do this graphically? First start with a plot of the graph of $f(x)$ along with the line $y = x$, as shown in the figure on the next page. Now pick a starting point x_0 on the x-axis. Getting $x_1 = f(x_0)$ is easy: You simply find the point $(x_0, f(x_0))$ that has on the graph of $f(x)$ above (or below) the point $x = x_0$. In other words, travel upward (or downward) from $(x_0, 0)$ to $(x_0, f(x_0))$. We now have $x_1 = f(x_0)$. How do we input x_1 into the "f machine" to find $x_2 = f(x_1)$? First, we have to find where x_1 lies on the x-axis. We do this by travelling from the point $(x_0, f(x_0))$ horizontally to the line $y = x$: the point of intersection will be $(f(x_0), f(x_0)) = (x_1, x_1)$. We are now sitting directly above (or below) the point $(x_1, 0)$ on the x-axis, which is patiently waiting to be input into $f(x)$ to produce $x_2 = f(x_1)$. We can travel from (x_1, x_1) to $x_1, 0)$ and then back up (or down) to $(x_1, f(x_1)) = (x_1, x_2)$. From here, we once again travel to the line $y = x$, intersecting it at (x_2, x_2). From here, we travel to the curve $y = f(x)$ to intersect it at (x_2, x_3), etc.. The procedure is illustrated below.



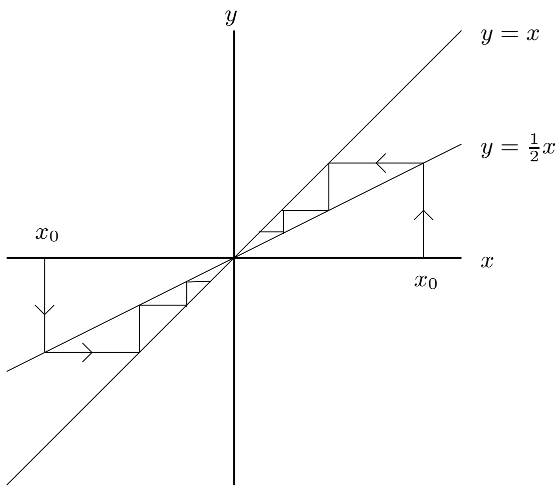
When you have performed this procedure a few times, you will see that including all the lines from intersection points (x_i, x_i) on the line $y = x$ and intersection points (x_i, x_{i+1}) on $y = f(x)$ to the x- and y-axis is unnecessary. In fact, these lines clutter up the figure. We have removed them to produce the figure below, in which the iteration process is presented in a much clearer way.



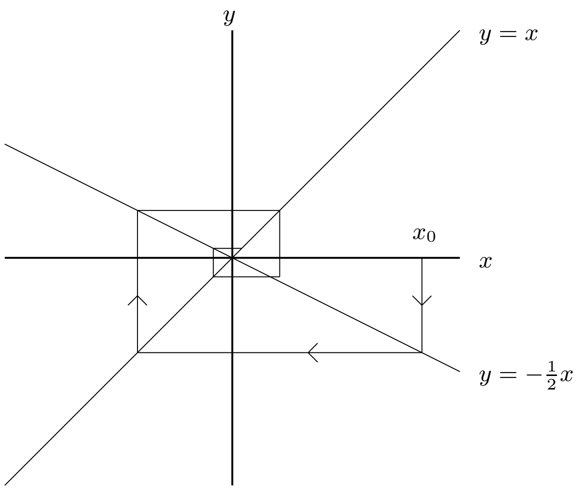
For rather obvious reasons, figures such as this one are called "cobweb diagrams". It appears as if the iterates x_0, x_1, x_2, x_3 are jumping back and forth, yet "zeroing in" on the point $(\bar{x}, \bar{x})$ at which the graph of f intersects the line $y = x$. Such a point of intersection must be a fixed point of f since it implies that $f(\bar{x}) = \bar{x}$. Of course, not all fixed points are attractive as the one in this diagram: The graphical procedure outlined above will give us some insight into what makes a fixed point attractive or repulsive.

We now examine some simple dynamical systems using this graphical approach.

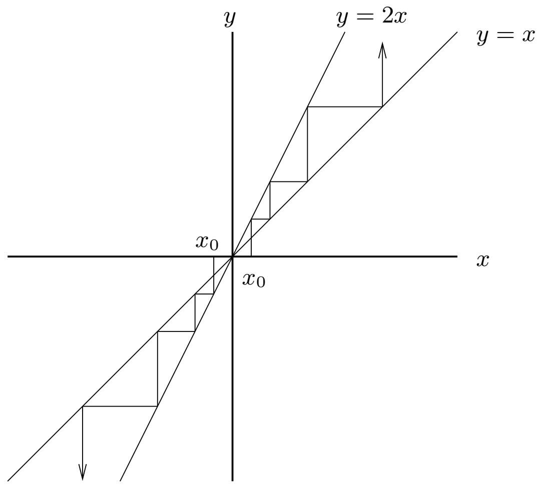
Example 1: $x_{n+1} = ax_n$, $0 < a < 1$. For $x_0 > 1$ or $x_0 < 1$, the graphical method shows the monotonic approach of the $x_n = a^n x_0$ toward the fixed point $x = 0$:



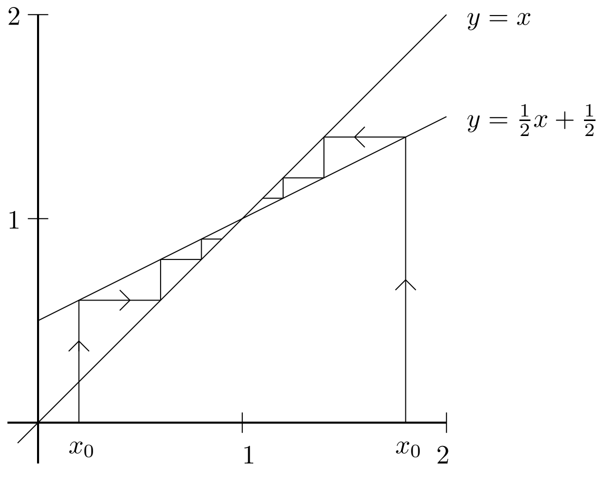
Example 2: $x_{n+1} = ax_n$, $-1 < a < 0$. The iterates x_n oscillate about $x = 0$, $x_n = (-1)^n |c|^n x_0$, with $x_n \to 0$ as $n \to \infty$:



Example 3: $x_{n+1} = ax_n$, $a > 1$. Here, $x_n = c^n x_0$. The iterates travel away from the fixed point 0:



Example 4: $x_{n+1} = ax_n + b$, $0 < a < 1$, $d > 0$

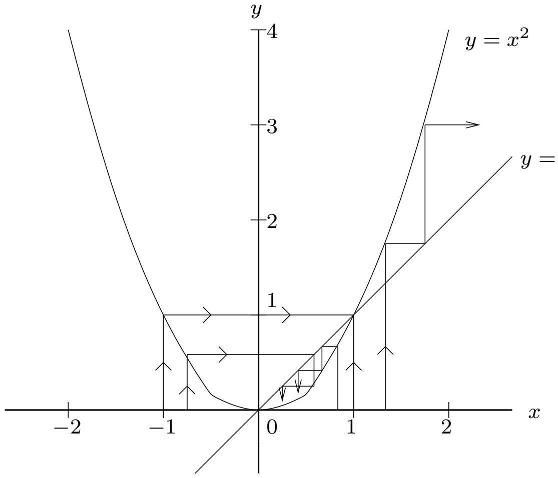


In this example, the two straight lines, $y = x$ and $y = cx + d$, can intersect at only one point, the fixed point of $f(x) = cx + d$, which is $\bar{x} = \frac{d}{1-c}$. Since $|c| < 1$, this fixed point is *attractive*: For any

$x_0 \in \mathbb{R}$, $x_n \to \bar{x}$ as $n \to \infty$. Note that this picture looks like that of Example 1, with the fixed point translated from $x = 0$ to $x = \frac{d}{1-c}$. The reader is once again encourage to examine the iteration of $f(x) = cx + d$ for the cases i) $-1 < c < 0$, ii) $c = -1$ and iii) $c < -1$.

Finally, we examine a couple of simple nonlinear iteration processes using the graphical method described above. The graphs of nonlinear functions $f(x)$ are not necessarily straight lines – in other words, they can "bend". As such $f(x)$ can have more than one fixed point. Each fixed point $\bar{x}_i$ can behave differently, i.e. one repulsive, the other attractive, so that the dynamics of the iteration process $x_{n+1} = f(x_n)$ can be quite complicated.

Example 5: $f(x) = x^2$, i.e. $x_{n+1} = x_n^2$.



Clearly, the fixed points of $f(x)$ are $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. Thus, if $x_0 = 0$, then $x_n = 0$. If $x_0 = 1$, then $x_n = 1$. If we begin with a point $x_0 \in (0, 1)$, then $x_1 = x_0^2 < x_0$. Likewise $x_0 > x_1 > x_2, \ldots, x_n > x_{n+1}, \ldots$ and $x_n \to 0$ as $n \to \infty$. (In other words, if we keep squaring a number starting in (0,1), we approach zero.)

If $x_0 \in (-1, 0)$, then $x_1 = x_0^2 \in (0, 1)$ and the above process is repeated, i.e. $x_n \to 0$ as $n \to \infty$. If $x_0 = 1$, then $x_n = 1$. for $n \geq 1$. If $x_0 > 1$, then $x_1 = x_0^2 > x_0$ and $x_{n+1} > x_n$, so that $x_n \to +\infty$ as $n \to \infty$. If $x_0 < -1$, then $x_1 = x_0^2 > 1$ and $x_n \to +\infty$ as $n \to \infty$.

A few other observations:

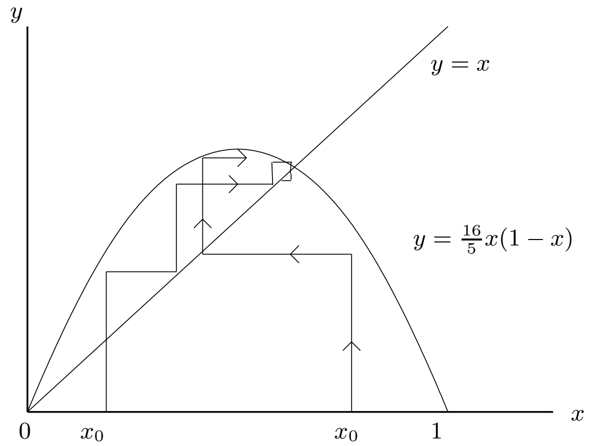
1. For all points $x_0 \in (-1, 1)$, $x_n \to 0$. We say that $\bar{x}_1 = 0$ is an *attractive* fixed point and that the interval $I = (-1, 1)$ is its *basin of attraction*.

2. The fixed point $\bar{x}_2 = 1$ is *repulsive* since all points in a neighbourhood of $\bar{x}_2$, $J = (1-\epsilon, 1+\epsilon)$ leave J after a finite number of iterations. (Those to the left of 1 go to 0; those to the right of 1 go to $+\infty$.)

3. The point $x = -1$ is mapped to the fixed point $\bar{x}_2 = 1$, and is called a **preperiodic point**, since it remains at $\bar{x}_2 = 1$ in future iterations.

4. For all other initial conditions, $x_0 \in (-\infty, -1) \cup (1, \infty) = \mathbb{R} - [0,1]$, the iterates $x_n \to +\infty$ as $n \to \infty$.

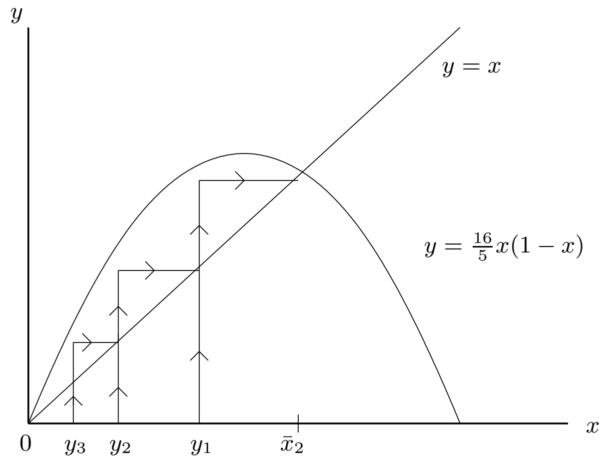
Example 6: $f(x) = \frac{16}{5}x(1-x)$, $x \in [0,1]$



f has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{11}{16}$ (Exercise). Clearly, if $x_0 = 0$, then $x_n = 0$. If x_0 is near 0, then the first few iterates x_1, x_2 are observed to travel away from 0 toward fixed point $\bar{x}_2$. The points $(x_n, f(x_n))$ travel "up the hump" until they reach roughly the height $\bar{x}_2$, whereupon they are directed toward the right side of the "hump". Graphical methods are insufficient to account for the long term behaviour of the x_n. If we magnify the graph of $f(x)$ near its fixed point $(\bar{x}_2, \bar{x}_2) = \left(\frac{11}{16}, \frac{11}{16}\right)$ and examine the "cobweb nature" of the iteration procedure $x_{n+1} = f(x_n)$, we find that a point x_0 near $\bar{x}_2 = \frac{11}{16}$ is sent by the function f to a point $x_1 = f(x_0)$ on the other side of $\bar{x}_2 = \frac{11}{16}$ and farther away from it, i.e. $|x_1 - \bar{x}_2| > |x_0 - \bar{x}_2|$. In other words, $\bar{x}_2 = \frac{11}{16}$ is a repulsive fixed point: If we choose $x_0 = \bar{x}_2$, then $x_n = \bar{x}_2$, i.e. we remain at $\bar{x}_2$. However, if we choose an x_0 "ϵ-close" to $\bar{x}_2$, with $\epsilon > 0$, the iterates x_n travel away from $\bar{x}_2$, no matter how small an ϵ is chosen.

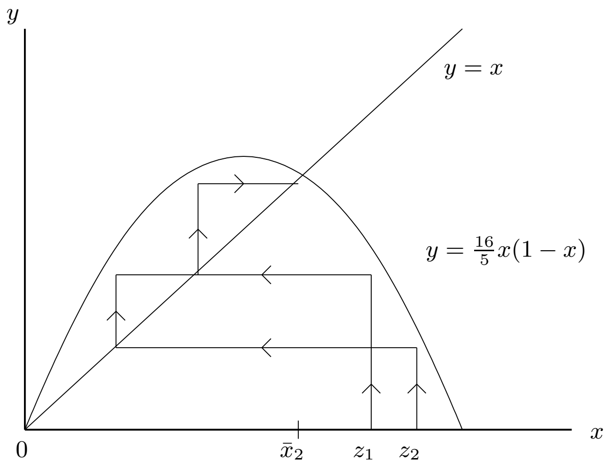
So what happens to the iterates x_n? Numerically, we find that they approach the two-cycle $(p_1, p_2) \cong (0.799, 0.513)$ as $n \to \infty$.

This is not the end of the story for this example. Even though the fixed point $\bar{x}_2$ is unstable, there is still an infinity of points that get mapped to $\bar{x}_2$ after a finite number of iterations and then stay at $\bar{x}_2$. To determine these points, we simply iterate "backwards", i.e. find graphically the point y_1 such that $f(y_1) = \bar{x}_2$, then find graphically the point y_2 such that $f(y_2) = y_1$ so that $f^2(y_2) = \bar{x}_2$, etc. In this way, we find an infinite set of **preperiodic** points y_n, $n = 1, 2, 3, \ldots$ such that $f^n(y_n) = \bar{x}_2$. Note that $y_n \to 0$ as $n \to \infty$.



In other words, these points can be found at arbitrarily small distances from the (repulsive) fixed point $\bar{x}_1 = 0$.

But that's not all! There's an infinity of points on the other side of the hump near $x = 1$ that also get mapped to $\bar{x}_2$ after a finite number of iterations. We can find them by going "right" instead of left in our "backwards" iteration procedure pictured above. In the figure below, we have found graphically the point z_1 near $n = 1$ such that $f(z_1) = y_1$ so that $f^2(z_1) = \bar{x}_2$ as well as the point z_2 such that $f(z_2) = y_2$ so that $f^3(z_2) = \bar{x}_2$. In this way, we find another infinite set of **preperiodic** points z_n, $n = 1, 2, 3, \ldots$ such that $f^{n+1}(z_n) = \bar{x}_2$. Note also that these points $z_n \to 1$ as $n \to \infty$.



APPENDIX: Material presented in Monday, February 27, 2017 tutorial

We are back to material that is most definitely a part of the course. The material presented in this tutorial is quite relevant to some problems in Assignment No. 6.

An "$N(\epsilon)$ problem"

Consider the sequence

$$a_n = 2 - \frac{1}{n^2}\,, \quad n \neq 1\,. \tag{45}$$

We can easily see that this sequence has a limit,

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(2 - \frac{1}{n^2}\right) \\ &= \lim_{n\to\infty} 2 - \lim_{n\to\infty}\frac{1}{n^2} \\ &= 2 - 0 \\ &= 2\,.\end{aligned} \tag{46}$$

Let's now prove this result, i.e.,

$$\lim_{n\to\infty} a_n = 2\,, \tag{47}$$

using the ϵ-$N(\epsilon)$ definition of the limit of a sequence. The above statement means that for any $\epsilon > 0$, there exists an $N(\epsilon) > 0$ such that

$$|a_n - 2| < \epsilon \quad \text{for all} \quad n > N(\epsilon)\,. \tag{48}$$

Our goal is to find $N(\epsilon)$ as a function of ϵ. From the definition of the sequence in Eq. (45),

$$a_n - 2 = -\frac{1}{n^2}\,, \tag{49}$$

which implies that

$$|a_n - 2| = \frac{1}{n^2}\,. \tag{50}$$

From Eq. (48), given an $\epsilon > 0$, we must find the n values for which

$$\frac{1}{n^2} < \epsilon\,. \tag{51}$$

We can multiply both sides of this inequality by n^2 and then divide both sides by $\epsilon > 0$ to yield the inequality,

$$n^2 > \frac{1}{\epsilon} \quad \Longrightarrow \quad n > \frac{1}{\sqrt{\epsilon}}\,. \tag{52}$$

Technically speaking, the ratio $\frac{1}{\sqrt{\epsilon}}$ may not be an integer. But we can take its integer part and then add 1 (or any number greater than 1), i.e., let

$$N(\epsilon) = \text{int}\left[\frac{1}{\sqrt{\epsilon}}\right] + 1\,, \tag{53}$$

where "int" denotes "integer part of". (For example, int(5.127) = 5.) We have therefore found the required $N(\epsilon)$, thus proving that the limit exists.

A "monotone sequence problem"

This is Question 81 of Stewart (Eighth Edition), p. 705:

> Show that the sequence,
>
> $$a_1 = 1\,, \quad a_{n+1} = 3 - \frac{1}{a_n}\,, \tag{54}$$
>
> is increasing and $a_n < 3$. Deduce that the sequence $\{a_n\}$ is convergent and find its limit.

Step 1: Show that the sequence is increasing. We see that

$$a_1 = 1\,, \quad a_2 = 2\,, \quad a_3 = \frac{5}{2}\,. \tag{55}$$

It seems as if the a_n are increasing. We'll prove it by induction. Assume that $a_n < a_{n+1}$ is true for $n = 1, 2, \cdots, N$. We must now show that it is true for $n = N + 1$, i.e., that $a_{N+1} < a_{N+2}$. Let's examine a_{N+2}:

$$a_{N+2} = 3 - \frac{1}{a_{N+1}}\,. \tag{56}$$

But $a_{N+1} > a_N > \cdots > a_1 > 0$ (by assumption). Therefore, taking reciprocals,

$$\frac{1}{a_{N+1}} < \frac{1}{a_N} \quad \text{(note that this is possible because the } a_n > 0)\,. \tag{57}$$

Multiply both sides by -1, which flips the inequality, i.e.,

$$-\frac{1}{a_{N+1}} > -\frac{1}{a_N}\,. \tag{58}$$

Now add 3 to each side,

$$3 - \frac{1}{a_{N+1}} > 3 - \frac{1}{a_N}\,. \tag{59}$$

But by definition of the sequence, this implies that

$$a_{N+2} > a_{N+1}\,. \tag{60}$$

The increasing property $a_n < a_{n+1}$ therefore holds for all $n \geq 1$.

Here is an alternate proof by induction. Once again, we assume that $a_n < a_{n+1}$ for $n = 1, 2, \cdots, N$ and attempt to show that it is true for $n = N+1$, i.e., that $a_{N+1} < a_{N+2}$. We examine the difference,

$$\begin{aligned} a_{N+2} - a_{N+1} &= \left(3 - \frac{1}{a_{N+1}}\right) - \left(3 - \frac{1}{a_N}\right) \\ &= -\frac{1}{a_{N+1}} + \frac{1}{a_N} \\ &= -\frac{a_N - a_{N+1}}{a_N a_{N+1}}\,. \end{aligned} \tag{61}$$

The denominator is positive since $0 < a_1 < a_2 \cdots a_N < a_{N+1}$ (by earlier assumption). The numerator is negative since $a_N < a_{N+1}$ (also by earlier assumption). Therefore,

$$a_{N+2} - a_{N+1} > 0 \quad \Longrightarrow \quad a_{N+1} < a_{N+2}\,, \tag{62}$$

the desired result. By induction, $a_n < a_{n+1}$ for all $n \geq 1$. The sequence is therefore increasing.

Step 2: Show that $a_n < 3$ for all $n \geq 1$. Once again, we'll prove the desired result by induction. We see that $a_1 < 3$ and $a_2 < 3$. Now assume that $a_n < 3$ for $n = 1, 2, \cdots N$. We must show that $a_{N+1} < 3$. First of all,

$$a_{N+1} = 3 - \frac{1}{a_N}\,. \tag{63}$$

But $a_N < 3$, which implies that

$$\frac{1}{a_N} > \frac{1}{3}\,. \tag{64}$$

This, in turn, implies that

$$-\frac{1}{a_N} < -\frac{1}{3}\,. \tag{65}$$

Now add 3 to both sides,

$$3 - \frac{1}{a_N} < 3 - \frac{1}{3} < 3\,. \tag{66}$$

But the LHS of this inequality is a_{N+1} so we have arrived at the desired result,

$$a_{N+1} < 3\,. \tag{67}$$

Therefore, the inequality $a_n < 3$ is true for all $n \geq 1$.

From Step 1 and Step 2, we have shown that the sequence $\{a_n\}$ is (i) increasing and (ii) bounded above. We may now use the Monotone Bounded Sequence Theorem to conclude that this sequence is convergent, i.e., it has a limit L. We may find L from the recursion relation defining the sequence, i.e.,

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty}\left(3 - \frac{1}{a_n}\right)\\ &= \lim_{n\to\infty} 3 - \lim_{n\to\infty}\frac{1}{a_n}\\ &= 3 - \frac{1}{\lim_{n\to\infty} a_n}\,,\end{aligned} \tag{68}$$

This implies that the limit L satisfies the equation,

$$L = 3 - \frac{1}{L}\,. \tag{69}$$

Multiplication by L and rearranging yields the following quadratic equation in L,

$$L^2 - 3L + 1 = 0\,. \tag{70}$$

The roots of this quadratic are

$$L_1 = \frac{3}{2} + \frac{1}{2}\sqrt{5} \approx 2.618\,, \qquad L_2 = \frac{3}{2} - \frac{1}{2}\sqrt{5} \approx 0.382\,. \tag{71}$$

Recalling that $a_1 = 1$ and that the sequence $\{a_n\}$ is increasing, the second root L_2 is not feasible since it is less than a_1. The root L_1 is therefore the limit of the sequence.

Lecture 22

Series (cont'd)

Introduction

`Relevant section of Stewart: 11.2`

We now begin a very important section of the course – the study of **infinite series** or, simply, **series**. Given an infinite sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, we now consider the following summation,

$$a_1 + a_2 + a_3 + \cdots , \tag{72}$$

which is compactly expressed in terms of Σ-notation as follows,

$$\sum_{n=1}^{\infty} a_n \quad \text{or simply} \quad \sum a_n . \tag{73}$$

This lecture followed quite closely the presentation found in Section 11.2 of Stewart. As such, it will not be reproduced in its entirely. Only some main points are given below.

First of all, the infinite summation in (72) must be understood in the same way that we considered improper integrals defined over an infinite domain of integration, e.g., $[0, \infty)$, namely, in terms of limits of appropriate **truncations**. In the case of series, we define the **partial sums** associated with the series in (72) as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \end{aligned} \tag{74}$$

and, in general,

$$s_n = a_1 + a_2 + \cdots a_n = \sum_{k=1}^{n} a_k , \quad n \geq 1 . \tag{75}$$

These partial sums form a new sequence, $\{s_n\}_{n=1}^{\infty}$. Note that

$$s_n = s_{n-1} + a_n , \quad n \geq 2 . \tag{76}$$

Definition: If the sequence of partial sums $\{s_n\}$ is convergent and

$$\lim_{n\to\infty} s_n = s , \tag{77}$$

exists and is a finite real number, then the series $\sum a_n$ is **convergent** (or the **series converges**) and the limit s is the **sum** of the series. We then write that

$$\sum_{n=1}^{\infty} a_n = s\,. \tag{78}$$

If the sequence $\{s_n\}$ is divergent, then the series $\sum a_n$ is **divergent**.

Example 1: The series,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\,. \tag{79}$$

Here, the elements of the series are given by

$$a_n = \frac{1}{2^n}\,, \quad n \geq 1\,. \tag{80}$$

We examine the first few partial sums of the series:

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ s_4 &= s_3 + \frac{1}{16} = \frac{15}{16}\,. \end{aligned} \tag{81}$$

The reader may well see a pattern here, i.e.,

$$s_n = \frac{2^n - 1}{2^n}\,, \quad n \geq 1\,. \tag{82}$$

This result can be proved by induction, but we skip the proof. We now examine whether the limit of this sequence exists:

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{2^n - 1}{2^n} \\ &= \lim_{n\to\infty} \left[1 - \frac{1}{2^n}\right] \\ &= 1\,. \end{aligned} \tag{83}$$

Therefore the series converges and its sum is 1.

Example 2: The series,

$$1 - 1 + 1 - 1 + \cdots\,. \tag{84}$$

Here, the elements of the series are given by

$$a_n = (-1)^{n-1}\,, \quad n \geq 1\,. \tag{85}$$

The first few partial sums of the series are

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + 1 = 1 \\ s_4 &= 1 - 1 + 1 - 1 = 0\,. \end{aligned} \tag{86}$$

The pattern is straightforward:

$$s_{2n-1} = 1 \quad s_{2n} = 0\,, \quad n \geq 1\,. \tag{87}$$

The sequence of partial sums oscillates between 1 and 0. Clearly,

$$\lim_{n\to\infty} s_{2n-1} = 1 \quad \text{and} \quad \lim_{n\to\infty} s_{2n} = 0\,. \tag{88}$$

Since the two limits are unequal, $\lim\limits_{n\to\infty} s_n$ does not exist. Therefore, the series is divergent.

In class, we proved that the famous geometric series,

$$a + ar + ar^2 + \cdots = \sum_{n=1} ar^{n-1} \quad a \neq 0\,, \tag{89}$$

is convergent if $|r| < 1$, in which case its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}\,. \tag{90}$$

If $|r| \geq 1$, the geometric series is divergent.

A very trivial application of this result is to the series,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots\,. \tag{91}$$

This is a geometric series with $a = 1$ and $r = x$. As such, it is convergent for $|x| < 1$ in which case its sum is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,, \quad -1 < x < 1\,. \tag{92}$$

As mentioned in the lecture, when we see the variable "x", we immediately start to think about functions. In fact, the RHS of the above equation is a function of x, namely,

$$f(x) = \frac{1}{1-x}\,. \tag{93}$$

Later, we'll show that the series on the LHS is the **Taylor series** of $f(x)$ at $x = 0$.

In the lecture, we also proved that the famous **harmonic series**,

$$\sum_{n=1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \,, \tag{94}$$

is **divergent**.

We then proved the following important result:

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0\,. \tag{95}$$

This result "makes sense": In order for the sequence of partial sums $\{s_n\}$ to converge, the contributions a_n that you add to each partial sum s_n to get the next one, s_{n+1} should go to zero.

That being said, **it is not true that if** $\lim_{n\to\infty} a_n = 0$**, then the series** $\sum_{n=1}^{\infty} a_n$ **converges.** The harmonic series is an example of a series for which $a_n \to 0$ but the series is divergent. One should always remember this example.

The Theorem presented above is the basis for the following important test:

Divergence test: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Lecture 23

Series (cont'd)

At the start of the lecture, we proved one of the three results given in Stewart's textbook as Theorem 8 of Section 11.2 on Page 714:

Theorem: If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$ and $(a_n - c_n)$ and their sums are, respectively,

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n \,, \tag{96}$$

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \,, \tag{97}$$

and

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \,, \tag{98}$$

We'll prove the middle result. Let s_n and t_n, $n \geq 1$, denote the partial sums of the two series, i.e.,

$$s_n = \sum_{k=1}^{n} a_n \,, \qquad t_n = \sum_{k=1}^{n} b_n \,. \tag{99}$$

Then by hypothesis,

$$\lim_{n\to\infty} s_n = s \,, \qquad \lim_{n\to\infty} t_n = t \,, \tag{100}$$

where s and t denote the sums of the two respective series. Now let u_n denote the partial sums of the combined series, i.e.,

$$u_n = \sum_{k=1}^{n}(a_k + b_k) \,, \quad n \geq 1 \,. \tag{101}$$

We need to show that $\lim\limits_{n\to\infty} u_n$ exists and is finite. This is rather easy. From the above equation, since we are working with a finite number of terms,

$$u_n = s_n + t_n \,, \qquad n \geq 1 \,. \tag{102}$$

Taking limits,

$$\begin{aligned} \lim_{n\to\infty} u_n &= \lim_{n\to\infty} (s_n + t_n) \\ &= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n \\ &= s + t \,. \end{aligned} \tag{103}$$

The proof of the second result is complete.

Integral test for convergence/divergence of positive series

`Relevant section of Stewart:  11.3`

The lecture followed the discussion of Stewart rather closely and won't be reproduced in detail. Here, we summarize the important parts.

Example 1: We start, as does Stewart, with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,. \tag{104}$$

The series coefficients a_n are defined as

$$a_n = f(n) \quad \text{where} \quad f(x) = \frac{1}{x^2} > 0 \quad \text{for} \quad x \geq 1\,. \tag{105}$$

From Figure 1 in Stewart, Page 719, which shows the graph of $f(x)$ along with boxes of area a_2, a_3, etc., that lie under the graph, we note that the partial sum s_n of the series is bounded above as follows,

$$\begin{aligned} s_n &= 1 + \frac{1}{4} + \frac{1}{9} \cdots + \frac{1}{n^2} \\ &< 1 + \int_1^n \frac{1}{x^2}\,dx \\ &< 1 + \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\,dx \\ &= 1 + \int_1^\infty \frac{1}{x^2}\,dx \\ &= 1 + 1 \\ &= 2\,. \end{aligned} \tag{106}$$

This result implies that the sequence of partial sums $\{s_n\}$ is bounded above. Moreover, this sequence is an increasing sequence:

$$s_{n+1} = s_n + a_n \quad \Longrightarrow \quad s_{n+1} - s_n = a_n > 0\,, \tag{107}$$

which implies that

$$s_n < s_{n+1} \quad n \geq 1\,. \tag{108}$$

From the Bounded Monotone Sequence Theorem, the sequence $\{s_n\}$ is convergent, i.e., the limit

$$\lim_{n\to\infty} s_n = s \tag{109}$$

exists and is finite. This implies that the series in (104) converges.

Example 2: We then considered the series,

$$\sum_{n=1} \frac{1}{n}, \tag{110}$$

i.e., the harmonic series studied in the previous lecture. We know that this series diverges, but let's adapt our integral method to treat this problem. Analogous to what is presented in Figure 2 of Stewart, Page 720, for the series $\sum \frac{1}{\sqrt{n}}$, it is easy to show that the partial sums of the harmonic series are bounded below as follows,

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &> \int_1^{n+1} \frac{1}{x}\,dx \\ &= \ln(n+1) - \ln(1) \\ &= \ln(n+1)\,. \end{aligned} \tag{111}$$

Clearly, as $n \to \infty$, $s_n \to \infty$, which implies that the series diverges.

The methods in which the above examples were examined essentially comprise the proofs of the following important result:

The Integral Test (for series with positive terms): Suppose that $f(x)$ is a continuous, positive and decreasing function for $x \geq 1$ and let $a_n = f(n)$. Then:

1. If $\int_1^\infty f(x)\,dx$ is convergent, then the series $\sum_{n=1}^\infty a_n$ is convergent.

2. If $\int_1^\infty f(x)\,dx$ is divergent, then the series $\sum_{n=1}^\infty a_n$ is divergent.

These two results may be combined with the following compact statement:

> The series $\sum_{n=1} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent.

Example: The series $\sum_{n=1}^\infty \frac{1}{n^5+1}$. Here, $a_n = f(n)$ where $f(x) = \frac{1}{x^5+1}$. Clearly, $f(x)$ is continuous, positive and decreasing for $x \geq 1$. We simply need to determine whether or not the improper integral

$$\int_1^\infty \frac{1}{x^5+1}\,dx \tag{112}$$

is convergent. It is **not necessary** to evaluate the integral, but simply to determine whether or not it is convergent. This makes the task much easier. We know that

$$f(x) \leq g(x) = \frac{1}{x^5}, \quad x \geq 1, \tag{113}$$

and that the improper integral

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^5}\,dx \tag{114}$$

converges. (It is an integral of the form $I_p = \displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ with $p > 1$.) Therefore, the integral in (112) converges, implying that the series converges.

Convergence of $\frac{1}{n^p}$ series: We can use the integral test to determine the p values for which the following series,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{115}$$

converges. Here, $a_n = f(n)$, where $f(x) = \frac{1}{x^p}$. First of all, for $p \leq 0$, it is not the case that $\lim_{n\to\infty} a_n = 0$. By the Divergence Test, the series will therefore not converge for $p \leq 0$. We therefore need only consider the cases $p \geq 0$. The function $f(x)$ is continuous, positive and decreasing for $x \geq 1$ for $p > 0$. Earlier in the course, we showed that the improper integral,

$$I_p = \int_1^\infty \frac{1}{x^p}\,dx, \tag{116}$$

converges for $p > 1$ and diverges for $p \leq 1$. From the Integral Test, we conclude that the series in (115) converges for $p > 1$ and diverges for $p \leq 1$.

Remarks:

- As in the case of improper integrals, the above result shows that the series coefficients a_n must decay sufficiently rapidly as $n \to \infty$, i.e., faster than $\frac{1}{n}$, for the series to converge.
- It is the "infinite tail" of a series which determines whether or not it is convergent. As such, the condition that $f(x)$ be decreasing for $x \geq 1$ may be replaced by $x \geq N$. Likewise, the improper integrals starting at $x = 1$ in the Integral Test may be replaced by the integrals

$$\int_N^\infty f(x)\,dx. \tag{117}$$

Estimating the sum of a series with positive terms

Let $\sum a_n$ be a convergent series with positive terms, i.e., $a_n > 0$, $n \geq 1$. Furthermore, assume that

$$f(n) = a_n\,, \qquad n \geq 1\,, \tag{118}$$

and that $f(x)$ is decreasing for $x \geq 1$.¿ We can use integrals of $f(x)$ to estimate how well the partial sums s_n of this series are approximating the true sum s.

Recall that the partial sums are defined as

$$s_n = a_1 + a_2 + \cdots + a_n\,, \quad n \geq 1\,, \tag{119}$$

and the sum of the series is

$$s = \lim_{n\to\infty} s_n = a_1 + a_2 + \cdots\,. \tag{120}$$

Now define the remainder R_n associated with the nth partial sum s_n as follows,

$$R_n = s - s_n\,, \quad n \geq 1\,. \tag{121}$$

It follows that

$$R_n = a_{n+1} + a_{n+2} + \cdots\,, \tag{122}$$

i.e., R_n is the "infinite tail" of the series which is dropped when we compute the partial sum s_n. From (121),

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty}(s - s_n) \\ &= s - \lim_{n\to\infty} s_n \\ &= s - s \\ &= 0\,. \end{aligned} \tag{123}$$

We can rewrite (121) as

$$s = s_n + R_n\,, \tag{124}$$

and consider R_n to be the **error in the approximation**

$$s \approx s_n\,. \tag{125}$$

Of course, this error goes to zero as $n \to \infty$. But can we estimate the error R_n for a given partial sum s_n, especially if we do not know the exact value of the sum s of the series?

The answer is "Yes." From Figure 3 on Page 723 of Stewart, using once again the integral of $f(x)$, it should be straightforward to see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)\,dx\,. \tag{126}$$

Similarily, from Figure 4 on Page 723 of Stewart,

$$R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_{n+1}^\infty f(x)\,dx\,. \tag{127}$$

We can combine these two results to write

$$\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx\,. \tag{128}$$

We can also use these results, in particular, the first result, in order to determine how many terms of a series must be used, i.e., how large n must be, so that the accuracy in approximating the true sum s with the partial sum s_n is less than a desired amount.

Example: Consider the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$, which was shown earlier in this lecture to converge. From our previous discussion, the remainder R_n associated with the partial sum,

$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}\,, \tag{129}$$

is bounded above as follows,

$$\begin{aligned} R_n &\le \int_n^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \int_n^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \left[\frac{1}{n} - \frac{1}{b}\right] \\ &= \frac{1}{n}\,. \end{aligned} \tag{130}$$

Clearly, $R_n \to 0$ as $n \to \infty$ but perhaps not very quickly.

To illustrate, we compute the partial sum s_{10} to be

$$s_{10} \approx 1.54977\ldots\,. \tag{131}$$

The error R_n associated with this approximation is then bounded as

$$R_{10} \leq \frac{1}{10}\,. \tag{132}$$

This means that

$$s - s_{10} = R_{10} \leq \frac{1}{10}\,, \tag{133}$$

which means that

$$s \leq s_{10} + \frac{1}{10} = 1.64977\ldots\,. \tag{134}$$

The actual value of the sum s is, to five decimal digits,

$$s = 1.644934\cdots\,. \tag{135}$$

We see that the actual error is

$$s - s_{10} \approx 0.095\,, \tag{136}$$

which is close to the estimated error.

The partial sum s_{100} is computed to be

$$s_{100} \approx 1.63498\cdots\,. \tag{137}$$

The error R_n associated with this approximation is then bounded as

$$R_{100} \leq \frac{1}{100}\,. \tag{138}$$

The actual error is

$$s - s_{100} \approx 0.00995\,, \tag{139}$$

which is very close to the estimated error.

One can still do better in obtaining estimates of the sum s from the partial sums, and we'll discuss this in the next lecture.