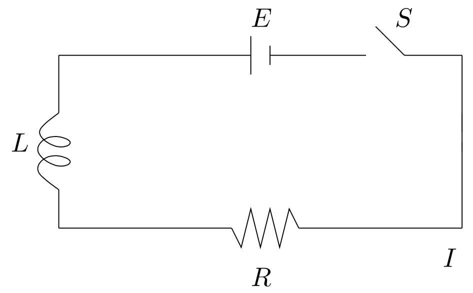
Lecture 18

Applications of differential equations (cont'd)

Electric circuits (cont'd)

Special case: LR circuit

Continuing with our discussion of simple electric circuits from the previous lecture, we now consider a special case in which the electric circuit is composed of a battery/generator, an inductor L and a resistor R, as sketched schematically below.



As mentioned in the previous lecture, Kirchhoff's Law for this circuit will be

$$L\frac{dI}{dt} + RI = E(t)\,. \tag{1}$$

This is a linear first-order DE in the current $I(t)$. We shall consider two particular cases for this circuit:

Case No. 1: A constant voltage, i.e.,

$$E(t) = E_0\,, \quad \text{constant.} \tag{2}$$

This corresponds to a direct-current (DC) power source. The DE in Eq. (1) then becomes

$$L\frac{dI}{dt} + RI = E_0\,. \tag{3}$$

This DE is both (i) separable and (ii) first-order linear. We'll solve it as a first-order linear DE. First, write it in standard form,

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L},. \tag{4}$$

But before solving this DE, we'll perform a little "qualitative analysis" in order to deduce the behaviour of solutions. We'll see that, once again, a good deal of information will be extracted. First, we'll rearrange the above DE as follows,

$$\frac{dI}{dt} = \frac{R}{L}\left(\frac{E_0}{R} - I\right). \tag{5}$$

As we would have done if solving this DE as a separable DE, we'll check if there are any constant solutions $I(t) = I$ for which the RHS is zero. Remember that this is the spirit of "qualitative analysis." Indeed, the constant function,

$$I(t) = I_0 = \frac{E_0}{R}, \tag{6}$$

is a solution of the DE: The RHS of Eq. (5) is zero and the LHS is a time derivative of a constant function, which is zero. This solution is also known as an **equilibrium solution** of the DE. (In fact, it is the only equilibrium solution.) In fact, this solution is in accordance with Ohm's Law: If we rewrite the above equation as

$$E_0 = I_0 R, \tag{7}$$

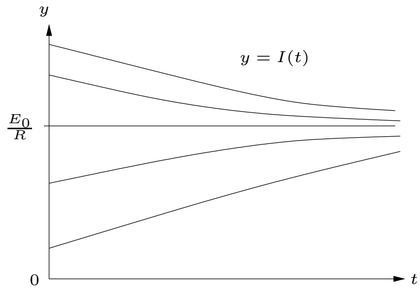
we see that the current I_0 is determined from the resistance R of the resistor and the voltage E_0. It is as if the inductor L is not present! But recall that the inductor is sensitive to **changes** in current. Continuing our qualitative analysis of Eq. (5), we note the following:

1. If, at any time $t \geq 0$, $I(t) < \frac{E_0}{R}$, then the RHS is positive, which implies the $\frac{dI}{dt} > 0$ which, in turn, implies that $I(t)$ is increasing. As it increases toward the value $\frac{E_0}{R}$, the RHS goes to zero, which implies that $I(t)$ is asymptotically approaching the constant value $\frac{E_0}{R}$ from below. (You saw this behaviour in the DE for the falling body in the presence of air resistance. The velocity $v(t)$ of the body was increasing toward the terminal velocity.)

2. If, at any time $t \geq 0$, $I(t) > \frac{E_0}{R}$, then the RHS is negative, which implies the $\frac{dI}{dt} < 0$ which, in turn, implies that $I(t)$ is decreasing. As it decreases toward the value $\frac{E_0}{R}$, the RHS goes to zero, which implies that $I(t)$ is asymptotically approaching the constant value $\frac{E_0}{R}$ from above. (You saw this behaviour in the DE for the falling body in the presence of air resistance. The velocity $v(t)$ of the body was increasing toward the terminal velocity.)

From this analysis, we can conclude that the solutions to the DE in Eq. (3) behave qualitatively as sketched below.

We'll now confirm these conclusions by solving the linear first-order DE in (4) exactly. The function $P(t) = \frac{R}{L}$ so the integrating factor of this DE is $e^{\frac{R}{L}t}$. (We won't refer to this integrating factor as $I(t)$



since $I(t)$ already denotes the current. Now multiply both sides of the DE by the integrating factor,

$$e^{\frac{R}{L}t}\frac{dI}{dt} + \frac{R}{L}e^{\frac{R}{L}t}I = \frac{E_0}{L}e^{\frac{R}{L}t} \,. \tag{8}$$

As should be the case, the LHS is an exact derivative,

$$\frac{d}{dt}\left[e^{\frac{R}{L}t}I\right] = \frac{E_0}{L}e^{\frac{R}{L}t} \,. \tag{9}$$

Now antidifferentiate w.r.t. t,

$$\begin{aligned} e^{\frac{R}{L}t}I &= \frac{E_0}{L}\int e^{\frac{R}{L}t}\,dt \\ &= \frac{E_0}{R}e^{\frac{R}{L}t} + C \,, \end{aligned} \tag{10}$$

so that we have

$$I(t) = \frac{E_0}{R} + Ce^{-\frac{R}{L}t} \,. \tag{11}$$

If we impose the initial condition $I(0) = I_0$,

$$I_0 = \frac{E_0}{R} + C \quad \Longrightarrow \quad C = I_0 - \frac{E_0}{R} \,, \tag{12}$$

so that the solutions become

$$I(t) = \frac{E_0}{R} + \left[I_0 - \frac{E_0}{R}\right]e^{-\frac{R}{L}t} \,. \tag{13}$$

We can see that for all solutions, i.e., all initial conditions $I_0 \geq 0$,

$$\lim_{t\to\infty} I(t) = \frac{E_0}{R} \,. \tag{14}$$

This is in agreement with the qualitative analysis performed earlier. The constant solution,

$$I(t) = \frac{E_0}{R} \,, \tag{15}$$

which we previously called the **equilibrium solution** of the DE, is also the **steady-state solution** of the LR circuit – all solutions $I(t)$ approach it as $t \to \infty$.

Case No. 2: An oscillatory and periodic EMF, i.e.,

$$E(t) = E_0 \cos \omega t \,, \tag{16}$$

where $\omega \geq 0$ is the angular frequency (radians/unit time) of the oscillation. This corresponds to an **alternating current** source with amplitude E_0.

Note: In class, we considered the case $E(t) = E_0 \sin \omega t$. The solution will not be too different – at least qualitatively – but they will differ slightly in terms of phase.

The first-order linear DE – in standard form – corresponding to this AC source will be

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E_0}{L} \cos \omega t \,. \tag{17}$$

Note that this DE is no longer separable – it will have to be solved as a linear first-order DE. Furthermore, because of the presence of the nonconstant $\cos \omega t$ term on the RHS, there are no constant (equilibrium) solutions.

The integrating factor of this DE will once again be $e^{\frac{R}{L}t}$. Multiplying the DE by this integrating factor, etc. will yield the expression,

$$I(t) = \frac{E_0}{L} \left(e^{-\frac{R}{L}t} \right) \int e^{\frac{R}{L}t} \cos \omega t \, dt + Ce^{-\frac{R}{L}t} \,. \tag{18}$$

We'll use the following result from the table of integrals (yes, you can use them now if you wish - otherwise the result can be derived with a little bit of work using integration by parts):

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[a \cos bx + b \sin bx) \right] \,. \tag{19}$$

The solution is then

$$I(t) = \frac{E_0}{L} \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2} \left[\frac{R}{L} \cos \omega t + \omega \sin \omega t \right] + Ce^{-\frac{R}{L}t} \,. \tag{20}$$

We'll modify the first part of the solution slightly to give

$$I(t) = \frac{E_0}{R^2 + (L\omega)^2} \left[R \cos \omega t + L\omega \sin \omega t \right] + Ce^{-\frac{R}{L}t} \,. \tag{21}$$

If we impose the initial condition $I(0) = I_0$, then

$$I_0 = \frac{E_0 R}{R^2 + (L\omega)^2} + C \,, \tag{22}$$

so that

$$C = I_0 - \frac{E_0 R}{R^2 + (L\omega)^2}\,, \tag{23}$$

in which case the solution becomes,

$$I(t) = \frac{E_0}{R^2 + (L\omega)^2}\left[R\cos\omega t + L\omega\sin\omega t\right] + \left[I_0 - \frac{E_0 R}{R^2 + (L\omega)^2}\right] e^{-\frac{R}{L}t}\,. \tag{24}$$

Perhaps the most important aspect of this solution, which could also have been deduced from the general solution in (21) is that as $t \to \infty$, the exponential $e^{-\frac{R}{L}t} \to 0$. As such, all solutions $I(t)$ approach the following function in the limit $t \to \infty$:

$$I_s(t) = \frac{E_0}{R^2 + (L\omega)^2}\left[R\cos\omega t + L\omega\sin\omega t\right]\,. \tag{25}$$

This is certainly not a constant solution but rather an oscillatory solution with frequency ω hence period $T = \dfrac{2\pi}{\omega}$ – this follows from the $\cos\omega t$ and $\sin\omega t$ functions, both of which have period T. The subscript "s" in "$I_s(t)$" is used to identify this function as the **steady state solution** to the DE in (17). It is essentially the **long time response** of the LR circuit to the EMF "forcing" function $E(t) = E_0\cos\omega t$. (You may have seen the idea of a steady state response to forcing in the case of the harmonic oscillator – in which a mass which is connected to a "Hookean" spring (restorative force $\mathbf{F}_{res} = -kx$) and moving on a surface that exerts a frictional force is subjected to an external forcing function.)

There is more to be learned from the steady state response function in Eq. (25). A linear combination of a sine and cosine function having the same frequency can always be written as a single sine or cosine function of the same frequency which is **phase-shifted**, i.e.,

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \phi) \quad \text{or} \quad C\sin(\omega t + \chi)\,. \tag{26}$$

Note that the amplitude C is the same in both phase-shifted functions, but the phases ϕ and χ are different - in fact, they are related to each other.

Here we shall derive the first result by means of a very simple "trick," rewriting the LHS as follows,

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\left[\frac{A}{\sqrt{A^2 + B^2}}\cos\omega t + \frac{B}{\sqrt{A^2 + B^2}}\sin\omega t\right]\,. \tag{27}$$

Note that the sums of the squares of the coefficients of the cosine and sine function inside the square brackets is 1. We'll now let ϕ be the angle such that

$$\cos\phi = \frac{A}{\sqrt{A^2 + B^2}}\,, \qquad \sin\phi = \frac{B}{\sqrt{A^2 + B^2}}\,. \tag{28}$$

Then the term in square brackets in (27) becomes

$$\cos\omega t\cos\phi + \sin\omega t\sin\phi = \cos(\omega t - \phi)\,, \tag{29}$$

where we have used the addition rule for the cosine function. Substitution into (27) yields the result,

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2+B^2}\cos(\omega t - \phi) \quad \text{where} \quad \tan\phi = \frac{B}{A}\,. \tag{30}$$

Note that we could also have let χ be the angle such that

$$\cos\phi = \frac{B}{\sqrt{A^2+B^2}}\,, \qquad \sin\phi = \frac{A}{\sqrt{A^2+B^2}}\,, \tag{31}$$

so that the term in square brackets in (27) becomes

$$\sin\omega t\cos\chi + \cos\omega t\sin\chi = \sin(\omega + \chi)\,. \tag{32}$$

We then have the result,

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2+B^2}\sin(\omega t + \chi) \quad \text{where} \quad \tan\chi = \frac{A}{B}\,. \tag{33}$$

A little work will show that the two phases, ϕ and χ, are related since the cos and sin functions are related to each other by a phase.

We'll use Eq. (30) to rewrite the steady state function $I_s(t)$ as follows,

$$\boxed{I_s(t) = \frac{E_0}{\sqrt{R^2+(L\omega)^2}}\cos(\omega t - \phi) \quad \text{where} \quad \tan\phi = \frac{L\omega}{R}\,.} \tag{34}$$

This equation tells us that the **response** of the LR circuit to the EMF forcing function,

$$E(t) = \cos(\omega t)\,, \tag{35}$$

is a **phase-shifted** $\cos\omega t$ function. (Physically, the response is "out-of-phase" with respect to the forcing – there is a time delay in the response.) The **amplitude** of the response $I_s(t)$,

$$A(\omega) = \frac{E_0}{\sqrt{R^2+(L\omega)^2}}\,, \tag{36}$$

is dependent upon the forcing frequency ω. Note that as ω decreases,

1. the amplitude $A(\omega)$ increases,

2. the phase shift ϕ decreases.

In the limit $\omega \to 0^+$, in which case the forcing function $E(t)$ becomes the constant function $E(t) = E_0$ (Case 1), the steady-state current becomes

$$I_s(t) = \frac{E_0}{R}\,, \tag{37}$$

which is in agreement with our result from Case 1.

Also note that as the forcing frequency ω increases, the amplitude function $A(\omega)$ decreases – in fact,

$$A(\omega) \to 0 \quad \text{as} \quad \omega \to 0^+\,. \tag{38}$$

(It is as if the forcing frequency is simply getting too large to produce a response in the system.)

Parametric representation of curves

`Relevant section from Stewart: 10.1`

Last week, we determined the trajectory of a projectile with mass m (e.g., a "cannonball") launched from position $(0,0)$ on the earth's surface with initial speed $v_0 > 0$ at an angle $0 < \theta\pi/2$ to the horizontal. Its trajectory is given as

$$\begin{aligned} x(t) &= (v_0 \cos\theta)\, t \\ y(t) &= (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 , \quad t > 0 . \end{aligned} \tag{39}$$

(Actually, the equations are valid only until the time t when the mass m hits the ground again at position $(R,0)$, where R is the range.) The above equations can be written compactly in vector form as follows,

$$\begin{aligned} \mathbf{x}(t) &= (x(t), y(t)) \\ &= \left((v_0 \cos\theta) t, (v_0 \sin\theta) t - \frac{1}{2} g t^2 \right) . \end{aligned} \tag{40}$$

The two equations in (39) comprise a **parametric representation** of the trajectory curve in $\mathbb{R}^2$. The word "parametric" refers to the parameter t – in this case, time – used to define the coordinates $x(t)$ and $y(t)$.

Recall that we then eliminated the t variable so that the curve could be expressed in the form $y = f(x)$. In this case,

$$y = Ax - Bx^2 , \tag{41}$$

where A and B are positive constants that depend upon the initial speed, v_0, the angle θ and the near-earth gravitational constant g. From this knowledge, one can deduce that the trajectory of the projectile lies on a parabola that opens up downward.

There are a couple of lessons to be learned from the above example:

1. The trajectory of the particle was determined by solving Newton's equations of motion – differential equations involving the horizontal and vertical position functions $x(t)$ and $y(t)$, respectively,

along with their derivatives. Fortunately, in this problem, $x(t)$ and $y(t)$ could be solved independently. The result: A parametric representation of the trajectory. Note that the natural parameter for such physical applications is **time**.

2. In order to get an idea of the nature of the trajectory in xy-space, we eliminated the parameter t to express y in terms of x. In other words, the curve is expressed in the form $y = f(x)$.

We now provide a more mathematical description of the parametric representation of curves. Furthermore, we'll make the discussion rather general to treat curves in arbitrary dimensions, i.e., $\mathbb{R}^n$, where $n \geq 2$, with the understanding that most applications will be concerned with $n = 2$ (motion in the plane) and $n = 3$ (motion in 3-space). We'll simply generalize the vector description of the trajectory curve in Eq. (40) as follows. We consider the following **vector-valued** function,

$$\mathbf{F}(t) = (f_1(t), f_2(t), \cdots, f_n(t)), \tag{42}$$

defined for some interval $a \leq t \leq b$. Each of the functions $f_k(t)$ is a mapping from $\mathbb{R} to \mathbb{R}$, i.e.,

$$f_k : \mathbb{R} \to \mathbb{R}, \quad 1 \leq k \leq n. \tag{43}$$

The result is that the function $\mathbf{F}(t)$ is a mapping from $\mathbb{R}$ to $\mathbb{R}^n$, i.e.,

$$\mathbf{F} : \mathbb{R} \to \mathbb{R}^n. \tag{44}$$

Translation:

- The **input** to $\mathbf{F}$ is a real number, i.e., $t \in [a, b]$.
- The **output** $bfF(t)$ is an n-vector, i.e., an element of $\mathbb{R}^n$.

In fact, if we consider the value $\mathbf{F}(t)$ to define the coordinates of a point $\mathbf{x}(t)$, then as we vary $t \in [a, b]$, the point

$$\mathbf{x}(t) = \mathbf{F}(t) = (f_1(t), f_2(t), \cdots, f_n(t)) \tag{45}$$

traces out a **path** or **curve** in $\mathbf{R}^n$. The coordinates of this point will be given by

$$x_k(t) = f_k(t), \quad 1 \leq k \leq n. \tag{46}$$

At this point, it is instructive to consider a few examples. (These examples were actually discussed in the Friday Lecture 20, but it is better, from a pedagogical perspective, to include them here.) In the examples below, which mostly involve curves in $\mathbb{R}^2$, we'll often use the notation

$$\mathbf{x}(t) = (x(t), y(t)) \tag{47}$$

instead of $(f_1(t), f_2(t))$. We also mention that the following notation,

$$\mathbf{x}(t) = (f(t), g(t)), \tag{48}$$

is used in the textbook by Stewart. Hopefully, there will be no confusion.

Example 1: The curve in $\mathbb{R}^2$ defined by

$$x(t) = x_0 + v_1 t \quad y(t) = y_0 + v_2 t\,, \tag{49}$$

where $t \in [0, 1]$. We note that

$$\mathbf{x}(0) = (x(0), y(0)) = (x_0, y_0)\,, \tag{50}$$

i.e., the curve starts at the point (x_0, y_0). Furthermore, it ends at the point,

$$\mathbf{x}(1) = (x(1), y(1)) = (x_0 + v_1, y_0 + v_2)\,. \tag{51}$$

Without loss of generality, we'll assume that v_1 and v_2 are both positive. If $\mathbf{x}(t)$ were describing the motion of a particle, then we see that the horizontal velocity of the particle is $x'(t) = v_1$ and its vertical velocity is $y'(t) = v_2$. These are constant, so it might be conjectured that the path of the particle, i.e., the curve $\mathbf{x}(t)$, is a straight line. In fact, when $t = 1$, the particle will have moved v_1 units to the right and v_2 units upward, suggesting that the slope of this line is $m = \frac{v_2}{v_1}$.

We'll confirm this result by computing the relationship between y and x. We'll do this by removing the parameter t. First step: Express t in terms of x.

$$t = \frac{x - x_0}{v_1}\,. \tag{52}$$

(Note that we have written x and not $x(t)$.) Second step: Substitute this result into the equation for $y(t)$.

$$\begin{aligned} y &= y_0 + v_2 \left(\frac{x - x_0}{v_1} \right) \\ &= \frac{v_2}{v_1} x + \left(y_0 - \frac{v_2}{v_1} x_0 \right)\,. \end{aligned} \tag{53}$$

This result is in the form $y = mx + b$, so we confirm that the slope of the curve is $m = \dfrac{v_2}{v_1}$.

Example 2: The curve in $\mathbb{R}^2$ defined by

$$x(t) = R\cos t \quad y(t) = R\sin t\,. \tag{54}$$

We could identify a few representative points but it is perhaps just as quick to note that

$$x(t)^2 + y(t)^2 = R^2 \cos^2 t + R^2 \sin^2 t = R^2 \,. \tag{55}$$

The trajectory $\mathbf{x}(t)$ is seen to lie on a circle of radius R centered at $(0, 0)$. In order to trace out this circle once, and only once, we restrict the t-interval to $[0, 2\pi]$.

If we restricted the t-interval to $[0, \pi]$, then the curve $\mathbf{x}(t)$ would be only the upper semicircle, i.e., $y(t) \geq 0$. On the other hand, if we were considering the above curve to represent the trajectory of a mass m rotating about the origin for all time $t \geq 0$, then we could consider the domain of definition to be $t \in [0, \infty)$.

Example 3: The curve in $\mathbb{R}^2$ defined by

$$x(t) = A\cos t \quad y(t) = B\sin t \,, \quad 0 \leq t \leq 2\pi \,, \tag{56}$$

where $A, B > 0$. In the special case, $A = B = R$, we have the previous example, i.e., a circle of radius R centered at $(0, 0)$. But in the case $A \neq B$, it seems that we have a "squished" circle. Note that

$$\frac{x}{A} = \cos t \quad \text{and} \quad \frac{y}{B} = \sin t \,, \tag{57}$$

so that

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \,, \tag{58}$$

which is the standard equation of an ellipse centered at $(0, 0)$ with x-intercepts $(A, 0)$ and $(-A, 0)$ and y-intercepts $(0, B)$, $(0, -B)$.

Example 4: The curve in $\mathbb{R}^3$ defined by

$$x(t) = R\cos t \quad y(t) = R\sin t \quad z(t) = At \,. \tag{59}$$

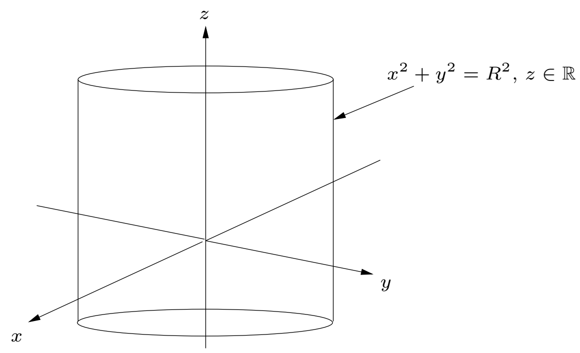
Note that as far as x and y are concerned, there is circular motion. As such, if you looked at the curve "from the top down," i.e., from the positive z-axis downward to the xy-plane, you would see only circular motion over the circle of Example 2.

But we now have another coordinate, i.e., $z(t) = At$. Let's assume that $A > 0$. Then the motion of the point in the z-direction is uniform, i.e., constant vertical velocity A.

The combination of circular motion in the xy direction and uniform motion in the z direction implies that the point traces out a **helix** in $\mathbb{R}^3$. This helix is a subset of the cylindrical surface, or simply "cylinder",

$$x^2 + y^2 = R^2, \quad z \in \mathbb{R}, \tag{60}$$

which is sketched below. The z-axis is the principal axis of this cylinder.



Derivative of the parametric representation function $\mathbf{F}(t)$.

In our discussion of the projectile motion problem, we worked with the following ideas without ever formally justifying them: If

$$\mathbf{x}(t) = (x(t), y(t)), \tag{61}$$

represented the position of the projectile, then

$$\mathbf{v}(t) = \mathbf{x}'(t) = (x'(t), y'(t)) \tag{62}$$

was its velocity at time t and

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t) = (x''(t), y''(t)) \tag{63}$$

was its acceleration. We now show these results more formally.

We first start with the vector valued function $\mathbf{F}(t)$,

$$\mathbf{F}(t) = (f_1(t), f_2(t), \cdots f_n(t)). \tag{64}$$

The formal derivative of this function with respect to t should be defined as follows:

$$\mathbf{F}'(t) = \lim_{h \to 0} \frac{\mathbf{F}(t+h) - \mathbf{F}(t)}{h}, \tag{65}$$

provided that the limit exists. Let's examine the Newton quotion on the RHS a little more closely. The numerator is the difference of two n-vectors, i.e.,

$$\begin{aligned}\mathbf{F}(t+h)-\mathbf{F}(t) &= (f_1(t+h), f_2(t+h), \cdots f_n(t+h)) - (f_1(t), f_2(t), \cdots, f_n(t)) \\ &= (f_1(t+h)-f_1(t), f_2(t+h)-f_2(t), \cdots f_n(t+h)-f_n(t)), \qquad (66)\end{aligned}$$

where the final line follows from the formal definition of the addition/subtraction of two n-vectors. Division by h in the Newton quotient is equivalent to multiplication by the scalar $\frac{1}{h}$ so that the Newton quotient becomes

$$\begin{aligned}\frac{\mathbf{F}(t+h)-\mathbf{F}(t)}{h} &= \frac{1}{h}\Big[(f_1(t+h), f_2(t+h), \cdots f_n(t+h)) - (f_1(t), f_2(t), \cdots, f_n(t))\Big] \\ &= \left(\frac{f_1(t+h)-f_1(t)}{h}, \frac{f_2(t+h)-f_2(t)}{h}, \cdots \frac{f_n(t+h)-f_n(t)}{h}\right), \qquad (67)\end{aligned}$$

We'll now take the limit as $h \to 0$:

$$\begin{aligned}\lim_{h\to 0}\frac{\mathbf{F}(t+h)-\mathbf{F}(t)}{h} &= \lim_{h\to 0}\frac{1}{h}\Big[(f_1(t+h), f_2(t+h), \cdots f_n(t+h)) - (f_1(t), f_2(t), \cdots, f_n(t))\Big] \\ &= \lim_{h\to 0}\left(\frac{f_1(t+h)-f_1(t)}{h}, \frac{f_2(t+h)-f_2(t)}{h}, \cdots \frac{f_n(t+h)-f_n(t)}{h}\right) \\ &= \left(\lim_{h\to 0}\frac{f_1(t+h)-f_1(t)}{h}, \lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h}, \cdots \lim_{h\to 0}\frac{f_n(t+h)-f_n(t)}{h}\right) \\ &= \big(f_1'(t), f_2'(t) \cdots, f_n'(t)\big), \qquad (68)\end{aligned}$$

provided that the individual limits exist. In summary, we have

$$\mathbf{F}'(t) = \big(f_1'(t), f_2'(t) \cdots, f_n'(t)\big), \qquad (69)$$

In other words, the derivative of the vector-valued function $\mathbf{F}(t)$ is the vector of derivatives of the component functions $f_i(t)$.

If we now interpret the curve,

$$\mathbf{x}(t) = \mathbf{F}(t), \qquad (70)$$

as the **position** of a particle at time t, then the derivative,

$$\mathbf{x}'(t) = \mathbf{F}'(t), \qquad (71)$$

represents is the **velocity**, $\mathbf{v}(t)$, of the particle. Note that the velocity is a **vector**, i.e., an element of $\mathbb{R}^n$. At a time t, it represents the instantaneous direction of motion of the particle.

Return to Example 2: The circle

$$x(t) = R\cos t\,, \quad y(t) = R\sin t\,, \tag{72}$$

which we can write compactly as

$$\mathbf{x}(t) = (x(t), y(t)) = (R\cos t, R\sin t)\,. \tag{73}$$

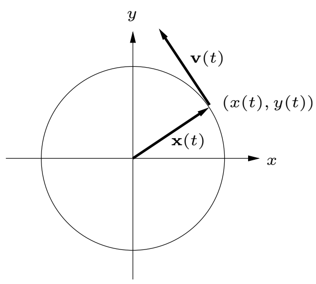
The velocity is easily computed to be

$$\mathbf{v}(t) = \mathbf{x}'(t) = (-R\sin t, R\cos t)\,. \tag{74}$$

At this point, we note the following: Taking the inner product of $\mathbf{x}(t)$ and $\mathbf{v}(t)$,

$$\begin{aligned} \mathbf{x}(t)\cdot\mathbf{v}(t) &= (R\cos t, R\sin t)\cdot(-R\sin t, R\cos t) \\ &= -R^2\cos t\sin t + R^2\sin t\cos t \\ &= 0\,, \quad \text{for all} \;\; t\,. \end{aligned} \tag{75}$$

This implies that the position vector $\mathbf{x}(t)$ and velocity vector $\mathbf{v}(t)$ are always orthogonal to each other. A representative situation is sketched below.



Note that we can place these vectors anywhere we wish, but it is convenient to place the position vector $\mathbf{x}(t)$ with head at (0,0) and tail at the point $(x(t), y(t)$. And it is convenient to place the velocity vector $\mathbf{v}(t)$ with head at point $(x(t), v(t))$.

What we have shown above is probably well known to you: The fact that the tangent to a circle at a point (x, y) on the circle is perpendicular to the radial line that extends from the center of the circle to point (x, y).

Now let's go one step further and compute the derivative of the velocity vector $\mathbf{v}(t)$ – in other words, the acceleration vector,

$$\mathbf{a}(t) = \mathbf{v}'(t)\,. \tag{76}$$

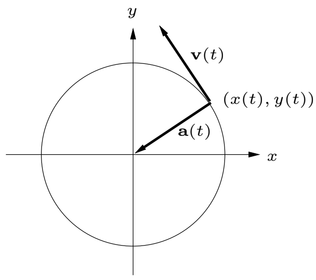
From Eq. (74),

$$\mathbf{v}'(t) = (-R\cos t, -R\sin t)\,. \tag{77}$$

But here we notice an interesting fact:

$$\mathbf{a}(t) = \mathbf{v}'(t) = -(R\cos t, R\sin t) = -\mathbf{x}(t)\,. \tag{78}$$

The acceleration vector points in the opposite direction to the position vector, as sketched below.



The fact that the acceleration vector points **inward** is consistent with the observation that the velocity vector $\mathbf{v}(t)$ is constantly moving leftward, i.e., inward. If $\mathbf{x}(t)$ represents the trajectory of a mass m moving along the circular orbit, then this trajectory would have to be caused by a **radial** force $\mathbf{F}$, i.e., a force was exerted on m toward the origin $(0,0)$. This is consistent with Newton's equation of motion, $\mathbf{F} = m\mathbf{a}$, since we have already determined that the acceleration vector is a radial vector that points inward, i.e., the vector $\mathbf{a}(t) = -\mathbf{x}(t)$.

Lecture 20

Parametric representation of curves (cont'd)

Computation of arclength

`Relevant section of Stewart:  10.2`

We now derive an integral formula for the arclength of the curve C obtained from the path,

$$\mathbf{x}(t) = (x(t), y(t)) \quad 1 \le t \le b\,, \tag{79}$$

where

$$x(t) = f(t) \quad y(t) = g(t)\,, \quad a \le t \le b\,. \tag{80}$$

We shall assume that $f(t)$ and $g(t)$ are differentiable on (a, b). We shall follow the same basic idea as was done for the arclength integral associated with the curve $y = f(x)$.

Following – what else? – the "Spirit of Calculus," we first construct a partition of the t-interval $[a, b]$ in the usual way, i.e., for a given $n > 0$, define $\Delta t = \dfrac{b-a}{n}$ and the partition points,

$$t_k = a + k\Delta t\,, \quad 0 \le k \le n\,, \tag{81}$$

so that $t_0 = a$ and $t_n = b$. Now let P_k denote the following points on the curve,

$$P_k = (x(t_k), y(t_k)) = (f(t_k), g(t_k)) \quad 0 \le k \le n\,. \tag{82}$$

(In the lecture, these points were called "Q_k".) These points P_k divide the curve C into n subcurves. We'll let C_k, $1 \le k \le n$, denote the subcurve with endpoints P_{k-1} and P_k. We'll also let L_k denote the length of C_k so that the length L of curve C is

$$L = \sum_{k=1}^{n} L_k\,. \tag{83}$$

Now comes the main idea: We approximate the length L_k of subcurve C_k by the straight line connecting P_{k-1} to P_k, i.e.,

$$L_k \simeq \|\overline{P_{k-1}P_k}\|\,. \tag{84}$$

For a diagram that illustrates this construction, please see Figure 4 of Stewart's textbook (Eighth Edition), Section 10.2, Page 652.

From Pythagoreas, the length of the line segment $\overline{P_{k-1}P_k}$ is

$$\begin{aligned} \|\overline{P_{k-1}P_k}\| &= \sqrt{(x(t_k) - x(t_{k-1})^2 + (y(t_k) - y(t_{k-1}))^2} \\ &= \sqrt{(f(t_k) - f(t_{k-1})^2 + (f(t_k) - f(t_{k-1}))^2} \\ &= \Delta t \sqrt{\left(\frac{f(t_k) - f(t_{k-1})}{\Delta t}\right)^2 + \left(\frac{f(t_k) - f(t_{k-1})}{\Delta t}\right)^2} \end{aligned} \tag{85}$$

By hypothesis, f and g are differentiable on (a, b). Therefore, by the Mean Value Theorem, there exist a $c_k \in [t_{k-1}, t_k]$ and $d_k \in [t_{k-1}, t_k]$ such that

$$f'(c_k) = \frac{f(t_k) - f(t_{k-1})}{\Delta t} \qquad g'(d_k) = \frac{g(t_k) - g(t_{k-1})}{\Delta t}. \tag{86}$$

(Note that c_k and d_k are not necessarily the same point, since we are dealing with two different functions, $f(t)$ and $g(t)$.) As a result, we have the approximation,

$$L_k \simeq \sqrt{(f'(c_k))^2 + (g'(d_k))^2}\,\Delta t\,. \tag{87}$$

The total length L of the curve is then approximated by the sum,

$$\begin{aligned} L &= \sum_{k=1}^{n} L_k \\ &\simeq \sum_{k=1}^{n} \sqrt{(f'(c_k))^2 + (g'(d_k))^2}\,\Delta t\,. \end{aligned} \tag{88}$$

This may be viewed as a generalized Riemann sum, since the points of evaluation c_k and d_k are not necessarily equal to each other. This is not a problem since, in the limit $n \to \Delta$, which implies that $\Delta x \to 0$, the above sum will converge (under the hypothesis that $f'(t)$ and $g'(t)$ are continuous functions) to the integral

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt\,.} \tag{89}$$

This is the formula for the arclength of the curve C traced out by the point,

$$\mathbf{x}(t) = (x(t), y(t) = (f(t), g(t))\,, \quad a \le t \le b\,. \tag{90}$$

There is one noteworthy point about the integral in (89): Its integrand is the length of the velocity vector,

$$\mathbf{v}(t) = \mathbf{x}'(t) = (f'(t), g'(t))\,, \tag{91}$$

i.e.,

$$L = \int_a^b \|\mathbf{x}'(t)\|\, dt\,. \tag{92}$$

But the length of the velocity vector, $\|\mathbf{v}(t)\|$, can be interpreted as the **speed** of the point as it traces the path/curve C. Let us now recall that the arclength formula may also be written as

$$L = \int_C ds\,. \tag{93}$$

In this case, we have

$$L = \int_C ds = \int_a^b \|\mathbf{x}'(t)\|\, dt\,, \tag{94}$$

which implies that the infinitesimal arclength ds at a time t is given by

$$ds = \|\mathbf{x}'(t)\|\, dt\,. \tag{95}$$

In terms of "noninfinitesimals," this implies that the distance Δs travelled over a small time interval Δt at time t is approximated as

$$\Delta s \simeq \|\mathbf{x}'(t)\|\, \Delta t\,. \tag{96}$$

In words,

distance travelled $\simeq$ speed $\times$ length of time interval.

Returning to the infinitesimal equation (95), we have

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\|\,. \tag{97}$$

The LHS does, in fact, represent speed – it is the instantaneous rate of change of arclength, or distance travelled, with respect to t.

These results connect quite well to our earlier results for the arclength of the curve $y = h(x)$ from $x = a$ to $x = b$. First of all, we may simply consider the following parametrization of this curve as follows,

$$x = f(t) = t\,, \quad y = h(x) = h(t)\,, \quad a \le t \le b\,, \tag{98}$$

so that the velocity vector is

$$\mathbf{x}'(t) = (1, h'(t))\,. \tag{99}$$

Eq. (89) then becomes

$$L = \int_a^b \sqrt{1 + (h'(t))^2}\, dt = \int_a^b \sqrt{1 + (h'(x))^2}\, dx\,, \tag{100}$$

which is the formula that we derived for the length of $y = h(x)$ earlier in the course.

So why all of the fuss about parametric representations of curves? The answer is that in our previous treatment, we had assumed that the curve was generated by a function $h(x)$. That meant that the curve could never cross itself or even have two different values of y at a given value of x, e.g., a circle. We don't have to worry about such things with parametrized curves. They can cross themselves, etc.. We simply "do the math".

Return to Example 2 from previous lecture: The following parametrization of a circle of radius R centered at the origin,

$$x(t) = R\cos t\,, \quad y(t) = R\sin t\,, \quad 0 \le t \le 2\pi\,. \tag{101}$$

The computation of the arclength is quite trivial, as compared to our earlier treatments. The velocity vector is

$$\mathbf{x}'(t) = (-R\sin t, R\cos t)\,, \tag{102}$$

which implies that

$$\|\mathbf{x}'(t)\| = R\,. \tag{103}$$

In other words, the speed is constant. The arclength L of the circle is therefore

$$\begin{aligned} L &= \int_a^b \|\mathbf{x}'(t)\|\,dt \\ &= \int_0^{2\pi} R\,dt \\ &= 2\pi R\,. \end{aligned} \tag{104}$$

Return to Example 3 from previous lecture: The following parametrization of an ellipse with axis lengths A and B and centered at the origin,

$$x(t) = A\cos t\,, \quad y(t) = B\sin t\,, \quad 0 \le t \le 2\pi\,. \tag{105}$$

The velocity vector is

$$\mathbf{x}'(t) = (-A\sin t, B\cos t)\,, \tag{106}$$

which implies that

$$\|\mathbf{x}'(t)\| = \sqrt{A^2\sin^2 t + B^2\cos^2 t}\,, \tag{107}$$

There is a complication, however, if $A \neq B$. Without loss of generality, let's assume that $A > B$. Then we can express the speed as

$$\begin{aligned} \|\mathbf{x}'(t)\| &= \sqrt{(A^2 - B^2)\sin^2 t + B^2 \sin^2 t + B^2 \cos^2 t} \\ &= \sqrt{B^2 + (A^2 - B^2)\sin^2 t} \\ &= B\sqrt{1 + r^2 \sin^2 t}\,, \end{aligned} \tag{108}$$

where

$$r = \sqrt{\frac{A^2 - B^2}{B^2}}\,. \tag{109}$$

The arclength integral for the ellipse is therefore

$$\begin{aligned} L &= \int_a^b \|\mathbf{x}'(t)\|\, dt \\ &= B \int_0^{2\pi} \sqrt{1 + r^2 \sin^2 t}\, dt\,. \end{aligned} \tag{110}$$

Unfortunately, there is no closed-form expression for this integral. It is an **elliptic integral** which must be evaluated in terms of an infinite series. Even if we consider the top half of the ellipse as the curve

$$y = b\sqrt{1 - \frac{x^2}{a^2}}\,, \quad -a \leq x \leq b\,, \tag{111}$$

we wouldn't be able to express the arclength in terms of an integral that could be evaluated in closed form. We would once again encounter an elliptic integral.

Extension of the above results to curves in $\mathbb{R}^n$

Here we simply mention that the arclength formula in Eq. (89) for curves in $\mathbb{R}^2$ can be generalized to curves in $\mathbb{R}^n$ given by the parametrization,

$$\mathbf{x}(t) = \mathbf{F}(t)\,, \quad a \leq t \leq b\,, \tag{112}$$

where

$$\mathbf{F}(t) = (f_1(t), f_2(t), \cdots, f_n(t))\,. \tag{113}$$

First of all, note that the velocity vector associated with the above trajectory is

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{x}'(t) \\ &= \mathbf{F}'(t) \\ &= (f_1'(t), f_2'(t), \cdots, f_n'(t))\,. \end{aligned} \tag{114}$$

The infinitesimal arc length ds on this path will be given by

$$\begin{aligned} ds &= \|bfx'(t)\|\,dt \\ &= \sqrt{(f_1'(t)^2 + (f_2'(t))^2 + \cdots + (f_n'(t))^2}\,dt\,. \end{aligned} \tag{115}$$

Then the length of the curve C generated by the parametrization in (112) is

$$\begin{aligned} L &= \int_C ds \\ &= \int_a^b \|\mathbf{x}'(t)\|\,dt \\ &= \int_a^b \sqrt{(f_1'(t)^2 + (f_2'(t))^2 + \cdots + (f_n'(t))^2}\,dt\,. \end{aligned} \tag{116}$$

This result is a natural extension of Eq. (89) for curves in the plane.

Indeed, it was never mention that in the special case $n = 1$, i.e., motion in one-dimension,

$$x(t) = f(t)\,, \quad a \le t \le b\,, \tag{117}$$

Eqs. (89) and (116) reduce to

$$L = \int_a^b |x'(t)|\,dt\,, \tag{118}$$

which we have encountered before. This may be interpreted as the **total length** travelled by a particle from time $t = a$ to time $t = b$ (as opposed to its net displacement).

Sequences

`Relevant section from Stewart: 11.1`

Only the last few minutes of the lecture were devoted to an introduction to sequences. The discussion followed very closely the presentation by Stewart in Section 11.1 and therefore will not be presented here.

Lecture 20

Sequences (cont'd)

`Relevant section from Stewart: 11.1`

The lecture once again followed very closedly the presentation by Stewart in Section 11.1 and will not be presented here. The lecture ended with the following definitions:

- increasing and decreasing sequences,
- sequences which are bounded from above,
- sequences which are bounded from below,
- bounded sequences – sequences which are both bounded from above and from below.

The next lecture will cover the so-called **Monotonic Sequence Theorem**.