Lecture 21

Linearization and linear approximation (cont'd)

Before going on to the next topic, let us return to the idea of using the linearization of a function to approximate the square root of a number. (Of course, we don't have to stop with square roots – may consider cube roots, fourth roots, even pth roots, where $0 < p < 1$.)

Let us stretch the idea from the previous lecture a little further. How would you use the linear approximation to approximate $\sqrt{92}$? As you may have concluded from the previous example, you express 92 as a deviation from a known square. There are at least two possibilities here since the closest squares are 81 and 100. There are others, but why go farther away from 92 if you don't have to? So we could consider the following two cases, recalling that our function $f(x) = \sqrt{1+x}$:

1. $\sqrt{92} = \sqrt{81+11} = 9\sqrt{1+\dfrac{11}{81}} = 9f\left(\dfrac{11}{81}\right) \approx 9L_0\left(\dfrac{11}{81}\right)$.

2. $\sqrt{92} = \sqrt{100-8} = 10\sqrt{1-\dfrac{8}{100}} = 10f\left(-\dfrac{8}{100}\right) \approx 10L_0\left(-\dfrac{8}{100}\right)$.

We suspect that the second approximation will be better since the x-value of $\frac{8}{100}$ lies closer to $a = 0$ than $\frac{11}{81}$ does. But let's go ahead and compute both approximations:

1. $9L_0\left(\dfrac{11}{81}\right) = 9\left(1+\dfrac{11}{162}\right) \approx 9.61111$.

2. $10L_0\left(-\dfrac{8}{100}\right) = 10\left(1-\dfrac{4}{100}\right) = 9.6$.

The actual value of $\sqrt{92}$ is, to five decimals, 9.59166. Therefore the errors of the above approximations are

1. Error $= 9.61111 - 9.59166 = 0.01945$,

2. Error $= 9.6 - 9.59166 = 0.00834$.

As conjectured, the second approximation is better.

Incremental form of the linear approximation and differentials

Recall that the linear approximation of a function f by its linearization at a takes the form,

$$f(x) \approx L_a(x) \quad \text{for } x \text{ near } a. \tag{1}$$

Substituting for $L_a(x)$, we have

$$f(x) \approx f(a) + f'(a)(x - a). \tag{2}$$

Let us now subtract $f(a)$ from both sides,

$$f(x) - f(a) \approx f'(a)(x - a). \tag{3}$$

The LHS of this expression is the change in f produced by the change in the independent varable in going from a to x. Because of this, Eq. (3) is known as the **incremental form** of the linear approximation of f at a. There is nothing new here – we're just looking at the linear approximation in another way. Eq. (3) may be written as follows,

$$\Delta y \approx f'(a)\Delta x, \qquad \Delta y = f(x) - f(a), \quad \Delta x = x - a. \tag{4}$$

In many applications, one is more interested in the **change** produced in a function as opposed to the **value** of the function itself.

We can rewrite Eq. (4) as follows,

$$\frac{\Delta y}{\Delta x} \approx f'(a), \tag{5}$$

which is nothing new to us. It is simply stating that the **Newton quotient**,

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a} \approx f'(a), \tag{6}$$

provides an approximation to $f'(a)$. And in the limit $x \to a$, the Newton quotient becomes the derivative, i.e.,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a), \tag{7}$$

as we well know.

We may also consider the so-called **infinitesimal version** of Eq. (4), written as

$$dy = f'(a)dx. \tag{8}$$

This result is true at all a, so we can write

$$dy = f'(x)dx. \tag{9}$$

There is a temptation to view this result as coming from the equation from the derivative, i.e.,

$$\frac{dy}{dx} = f'(x), \tag{10}$$

by simply multiplying both sides by dx, and cancelling on the LHS. Unfortunately, this "trick works" in the case of single-variable functions, but it doesn't in the case of functions of more than one variable.

The differential form in (8) or the next equation are to be interpreted as follows: it expresses the connection between the "infinitesimals" dy and dx. And infinitesimally small change dx will produce the infinitesimally small change dy – the "conversion factor" is the derivative $f'(x)$. When we move from these infinitesimally small changes to larger "macroscopic" ones, i.e., Δx and Δy, we then acknowledge that the equation involving differentials becomes an approximation, i.e.,

$$\Delta y \approx f'(a)\Delta x. \tag{11}$$

We now consider an example of the use of the incremental form of the linear approximation.

Example 1: Estimate the change in volume of a sphere of radius r due to a change Δr in its radius. First of all, the volume V of a sphere of radius r is given by

$$V(r) = \frac{4}{3}\pi r^3. \tag{12}$$

Its derivative with respect to r is given by

$$\frac{dV}{dr} = V'(r) = 4\pi r^2. \tag{13}$$

The incremental form of the linear approximation to V at r then follows,

$$\Delta V \approx V'(r)\Delta r. \tag{14}$$

This is the equation with which we shall work.

That being said, note that the above equation can also be seen to arise from the differential associated with $V(r)$, namely,

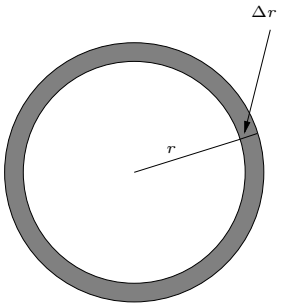
$$dV = 4\pi r^2 dr. \tag{15}$$

We can now change dV to ΔV and dr to Δr, also changing the equality to an approximation.

From Eq. (14) and the formula for the derivative $V'(r)$ in (13), we have

$$\Delta V \approx 4\pi r^2 \Delta r. \tag{16}$$

This is our desired result – the change in volume ΔV produced by a change in radius Δr. Note that the conversion factor $4\pi r^2$ is the **surface area** of a spherical surface of radius r. This can be interpreted geometrically. A very tiny change Δr to the radius r of the sphere produces a thin "onion skin" covering of the sphere, with inner radius r and outer radius $r + \Delta r$. A cross-section of this sphere and skin is shown below.



If we could now take a pair of scissors and cut this thin skin, open it up and flatten it, it would – at least schematically – be equivalent to a solid with top area $A = 4\pi r^2$ (the surface area of the skin) and thickness Δr. As such, its volume ΔV would be roughly the amount given in (16).

One final comment on this result – the **relative change** $\dfrac{\Delta V}{V}$ in the volume of the sphere produced by a change Δr in its radius is obtained from Eq. (16) by dividing by V:

$$\frac{\Delta V}{V} \approx \frac{4\pi r^2 \Delta r}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = 3\frac{\Delta r}{r}. \tag{17}$$

In other words, the relative change in volume is three times the relative change in the radius.

At this point, you may be wondering where the factor "3" on the RHS comes from. One might conjecture that it comes from the exponent "3" in the expression for the volume, Eq. (12). In fact, this is the origin of the factor, but it may not be so clear how it becomes a multiplicative factor in Eq. (17). The LHS of Eq. (17), $\dfrac{\Delta V}{V}$, looks like it could be related to the derivative of $\ln V$. And the term $\dfrac{\Delta r}{r}$ on the RHS of Eq. (17) looks like it could be related to the derivative of $\ln r$. If we now exponentiate, we have a relationship between V and r^3. This is not a "rigorous" argument, but it gives an idea of what is going on.

Example 2: Here is another example, motivated from science. Recall the **ideal gas equation** for the pressure, P, volume, V, and temperature, T, of a gas in a container.

$$PV = nRT. \tag{18}$$

Here, n is the number of moles of gas and R is the so-called **ideal gas constant**. If we consider T as fixed, then the above relation essentially becomes

$$PV = C, \qquad \text{constant}. \tag{19}$$

Thus, P and V are inversely proportional to each other: If we increase P, V will decrease, and *vice versa*. We now imagine that P and V are changing in time, while obeying Eq. (19). The question is: How exactly are their rates of change related?

We'll consider P as a function of V, i.e.,

$$P(V) = \frac{C}{V}, \tag{20}$$

which implies that

$$P'(V) = -\frac{C}{V^2}. \tag{21}$$

Now use the incremental form of the linear approximation,

$$\Delta P \approx P'(V)\Delta V, \tag{22}$$

so that

$$\Delta P \approx -\frac{C}{V^2}\Delta V. \tag{23}$$

This provides the relationship between change in pressure P and change in volume V. Clearly, ΔP and ΔV have different signs, as expected. But let us now divide both sides of the above result by P in an effort to arrive at a result that relates the relative change in P to the relative change in V:

$$\frac{\Delta P}{P} \approx -\frac{C}{PV^2}\Delta V. \tag{24}$$

There are a number of ways to proceed. One way is to employ Eq. (19) in the RHS of the above relation:

$$\frac{\Delta P}{P} \approx -\frac{C}{PV^2}\Delta V = -\frac{C}{(PV)V}\Delta V = -\frac{C}{CV}\Delta V = \frac{1}{V}\Delta V. \tag{25}$$

The net result is

$$\frac{\Delta P}{P} \approx -\frac{\Delta V}{V}. \tag{26}$$

In other words, the relative change in the pressure is minus the relative change in the volume. In differential form, the above becomes an equality,

$$\frac{dP}{P} = -\frac{dV}{V}. \tag{27}$$

Exercise: Find the equation relating the relative changes in P and V for the **nonideal gas equation**,

$$PV^\gamma = C, \tag{28}$$

where $\gamma > 0$ is a constant.

Related Rates

(Relevant section from Stewart, Seventh Edition: Section 3.9)

We shall treat this subject very briefly since many of you may have seen this in high school. Basically, a "related rate" problem is concerned with how the rate of change of one quantity is related to the rate of change of another quantity, when these two quantities are in some way "connected."

To illustrate, let us return to the volume of a sphere. Suppose that the radius r is changing in time, i.e., $r = r(t)$. Then the volume V of the sphere will also be changing in time. Mathematically,

$$V = V(t) = \frac{4}{3}\pi[r(t)]^3. \tag{29}$$

We now ask the question: How is $\dfrac{dV}{dt}$ related to $\dfrac{dr}{dt}$?

This is a "Chain Rule" problem. We acknowledge that V is a function of r which, in turn, is a function of t, so that

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}. \tag{30}$$

Since $V = \dfrac{4}{3}\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}. \tag{31}$$

This is the desired connection between the rate of change of the volume with the rate of change of the radius.

Note: The Chain Rule in Eq. (30) could also have been expressed in the form which you saw in high school:

$$\frac{dV}{dt} = V'(r(t))r'(t). \qquad \text{(Remember the Chain Rule formula } f'(g(x))g'(x)\text{?)} \tag{32}$$

Lecture 22

Some applications to classical mechanics

We now show how the concepts of Calculus examined to date are important in classical mechanics. Our study will be quite simple - we consider only one-dimensional motion of a mass. (Higher dimensional motion would require functions of several variables – you will study the Calculus of such functions at the end of MATH 138 and in your second-year Calculus courses.)

In what follows, we assume that a mass m is moving along the x-axis under the influence of a force $\mathbf{F} = f(x)\hat{\mathbf{i}}$. Note that the force may be dependent on the position x of the mass. Furthermore, note that if $f(x) > 0$, the force is acting in the positive x-direction and if $f(x) < 0$, the force is acting in the negative direction. If $f(x) = 0$, then there is no force acting on the mass at the position x. The trajectory of the mass will be determined by Newton's Second Law,

$$\mathbf{F} = m\mathbf{a}, \tag{33}$$

where $\mathbf{a}(t)$ denotes the acceleration (vector) of the mass at time t. We'll let $\mathbf{a}(t) = a(t)\,\mathbf{i}$ so that the above equation becomes, in vector form,

$$f(x)\,\hat{\mathbf{i}} = ma\,\hat{\mathbf{i}}. \tag{34}$$

Equating components of the two vectors is trivial, since there is only one component in each case. The result is

$$f(x) = ma(t). \tag{35}$$

This equation may be written as follows,

$$f(x) = m\frac{dv}{dt}, \tag{36}$$

where $v(t)$ denotes the velocity of the mass at time t. But we may go one step further and write

$$f(x) = m\frac{d^2x}{dt^2}, \tag{37}$$

where $x(t)$ denotes the position of the mass at time t. Eq. (37) is a second-order differential equation (DE) in the unknown function $x(t)$. In some special cases for the force $f(x)$, we may solve this equation

"in closed form," which means that the position function $x(t)$ can be expressed in terms of known functions such as $\sin t$, $\cos t$, powers of t, etc..

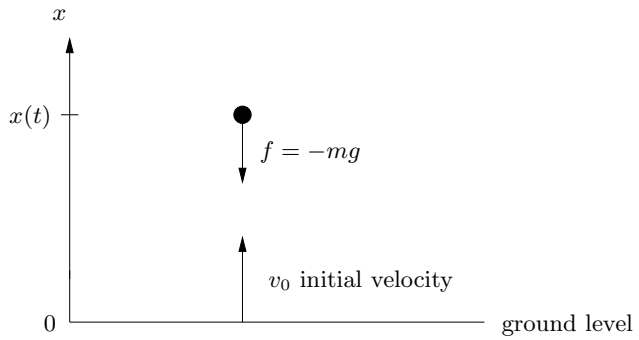
In a later course on DEs, you'll learn that there generally exists a family of solutions to the second-order DE in (37). If we impose two **initial conditions** on the solution, usually of the form

1. $x(0) = x_0$, the initial position,

2. $x'(0) = v_0$, the initial velocity,

a unique solution $x(t)$ satifying these conditions can be extracted. This will be shown in the examples that follow.

Example 1: The projectile problem. Here we consider a problem with which you are already familiar – launching a ball vertically upward from the ground with velocity $v_0 > 0$. In this problem, we assume that the position $x(t)$ represents the height of the ball. As such, the x-axis will be vertical, with the positive x-axis pointing vertically upward. We assume that v_0 is sufficiently small so that the ball stays relatively close to the earth – close enough that the gravitational force acting on the ball can be assumed to be $f(x) = -mg$, a constant downward force. We also assume that no other forces are acting on the ball, in particular, air resistance. (We'll discuss air resistance later.) The situation is sketched below.



Newton's Second Law in Eq. (35) becomes

$$f = -mg. \tag{38}$$

Since $f = ma = mx''(t)$, we have

$$m\frac{d^2x}{dt^2} = -mg, \tag{39}$$

or, cancelling out m from both sides,

$$\frac{d^2x}{dt^2} = -g. \tag{40}$$

This is the second-order differential equation in $x(t)$ associated with this problem. You are no doubt familiar with the solution of this problem, but let's proceed one step at a time.

We first reduce the order of the DE in (40) by using the velocity function $v(t) = x'(t)$ instead of $x(t)$. Eq. (40) becomes

$$\frac{dv}{dt} = -g. \tag{41}$$

This is a first-order DE in the function $v(t)$. Once again, you are familiar with the solution of this DE.

We are looking for a function $v(t)$, the derivative of which, with respect to t, is $-g$. By "inspection," we know that the function $v(t) = -gt$ satisfies (41). But you also know that we can add a constant to this function, i.e.,

$$v(t) = -gt + C_1. \tag{42}$$

When we take derivatives, we see that this function satisfies (41). But is it possible that there are other functions that satisfy (41)? We address this question below by discussing the concept of an *antiderivative*. We're actually doing this a little ahead of schedule (it's covered in Section 4.9 of Stewart's text) but the concept is necessary in our current applications to mechanics.

Antiderivatives

(Relevant section from Stewart, Seventh Edition: 4.9)

Definition: A function F such that $F'(x) = f(x)$ for all x (in its domain of definition) is called an **antiderivative** of f.

Examples:

1. $f(x) = x$. An antiderivative of x is $\frac{1}{2}x^2$. But so is $\frac{1}{2}x^2 + 10$. Or $\frac{1}{2}x^2 + C$, where C is any constant.

2. $f(x) = \sin x$. An antiderivative of $\sin x$ is $-\cos x$. But so is $-\cos x + C$ where, once again, C is any constant.

From the above examples, we see that if we have an antiderivative F of a function f, then we can add an arbitrary constant C to it and still have an antiderivative of f. But are there any other antiderivatives? The following result answers this question.

Theorem: Suppose that F_1 and F_2 are antiderivatives of a function f. Then

$$F_1(x) = F_2(x) + C \tag{43}$$

for some constant C.

We'll have to prove this theorem in the next lecture.

Lecture 23

Antiderivatives (cont'd)

(Relevant section from Stewart, Seventh Edition: 4.9)

We now prove the following theorem, introduced at the end of the previous lecture.

Theorem: Suppose that F_1 and F_2 are antiderivatives of a function f. Then

$$F_1(x) = F_2(x) + C \tag{44}$$

for some constant C.

Proof: From the assumptions, for all x in the domains of definition,

$$\begin{aligned} F_1'(x) &= f(x) \\ F_2'(x) &= f(x). \end{aligned} \tag{45}$$

Now subtract the second equation from the first:

$$F_1'(x) - F_2'(x) = 0. \tag{46}$$

But this implies that

$$\frac{d}{dx}[F_1(x) - F_2(x)] = 0. \tag{47}$$

Since this is true for all x, it follows that the LHS is a constant, i.e.,

$$F_1(x) - F_2(x) = C, \tag{48}$$

for some constant C. (This statement "seems" to be correct – we'll prove it rigorously in a few lectures with the help of the "Mean Value Theorem.") Therefore,

$$F_1(x) = F_2(x) + C, \tag{49}$$

and the theorem is proved.

A consequence of this theorem is that once we know a single antiderivative F of a function f, we can generate all others by simply adding a constant to $F(x)$.

We now return to our classical mechanics problem. Solving the DE

$$a(t) = \frac{dv}{dt} = -g \tag{50}$$

amounts to finding an antiderivative of the function $a(t) = -g$. An obvious antiderivative of $-g$ is $-gt$. From the previous theorem, it follows that all antiderivatives of $-gt$, *hence all solutions of the above DE*, have the form

$$v(t) = -gt + C_1, \tag{51}$$

where C_1 is an arbitary constant. We can remove the arbitrariness of the constant C_1 by imposing an initial condition on the function $v(t)$. In this case, we may impose the original **initial condition** of our problem on $v(t)$: At time $t = 0$, $v(0) = v_0$, the initial velocity at which the ball was launched. This implies that

$$v(0) = -g \cdot 0 + C_1 = v_0 \quad \Rightarrow \quad C_1 = v_0. \tag{52}$$

The solution is then

$$v(t) = v_0 - gt, \tag{53}$$

which is well known to you. A plot of $v(t)$ vs. t is a straight line with slope $-g$. It starts out at value $v_0 > 0$ at $t = 0$ and decreases to 0 at time $t_1 = \dfrac{v_0}{g}$. At this time, the ball has achieved its maximum height h and will now begin to descend to earth.

We may now proceed to solve for the position $x(t)$ of the ball from the equation

$$\frac{dx}{dt} = v_0 - gt. \tag{54}$$

Clearly, $x(t)$ is an antiderivative of the velocity function $v(t) = v_0 - gt$. By "inspection," we can find one antiderivative, i.e.,

$$x(t) = v_0 t - \frac{1}{2} g t^2. \tag{55}$$

From our earlier discussion of antiderivatives, this means that the set of all antiderivatives of $v_0 - gt$ is given by

$$x(t) = v_0 t - \frac{1}{2} g t^2 + C_2, \tag{56}$$

where C_2 is an arbitrary constant. Once again, we can remove the arbitrariness of the constant C_2 by imposing an initial condition on $x(t)$, say,

$$x(0) = x_0 = C_2 \quad \Rightarrow \quad C_2 = x_0. \tag{57}$$

Then the resulting solution is

$$x(t) = x_0 + v_0 t - \frac{1}{2} g t^2, \quad t \geq 0, \tag{58}$$

with which you are familiar.

Recall that $t_1 = \dfrac{v_0}{g}$ was the time at which the decreasing velocity $v(t)$ became zero, corresponding to maximum height h of the trajectory. We substitute this time into Eq. (58) to obtain

$$h = x(t_1) = x\left(\frac{v_0}{g}\right) = x_0 + \frac{v_0^2}{2g}. \tag{59}$$

In most discussions, the ball is launched from ground level, i.e., $x_0 = 0$.

Just to refresh our memory here: The function $x(t)$ in (58) is a solution of Newton's equation of motion,

$$\frac{d^2x}{dt^2} = -g, \tag{60}$$

(differentiate the solution twice to verify this) which satisfies the initial conditions,

$$x(0) = x_0, \quad x'(0) = v(0) = v_0. \tag{61}$$

Example 1(b): A minor variation on the above problem. We now consider the effect of **air resistance** in this problem. In addition to the gravitational force, which we'll now denote as $f_g = -mg$, there is a force that opposes the motion in either direction. A standard model for this resistive force which works quite well for small velocities is

$$f_{res} = -kv, \qquad k \geq 0, \quad \text{constant}. \tag{62}$$

Note:

1. When $v > 0$, i.e., the ball is moving upward, the force $f_{res} < 0$, implying that the force acts downward,

2. When $v < 0$, i.e., the ball is moving downward, the force $f_{res} > 0$, implying that the force acts upward,

Newton's equation for the projectile now becomes

$$ma = f_g + f_{res}. \tag{63}$$

In terms of $x(t)$, this equation becomes

$$m\frac{d^2x}{dt^2} = -mg - k\frac{dx}{dt}. \tag{64}$$

This is a second-order DE in $x(t)$. It is often written in the following more standard form,

$$\frac{d^2x}{dt^2} + \frac{k}{m}\frac{dx}{dt} = -g. \tag{65}$$

There are standard methods to solve this DE – you will learn about them in your second-year course on differential equations.

We can also step back a bit and rewrite Eq. (64) in terms of the velocity function $v(t) = x'(t)$:

$$m\frac{dv}{dt} = -mg - kv. \tag{66}$$

This is a first-order DE in $v(t)$, which is even easier to solve than (64). It is convenient to divide out the m from both sides,

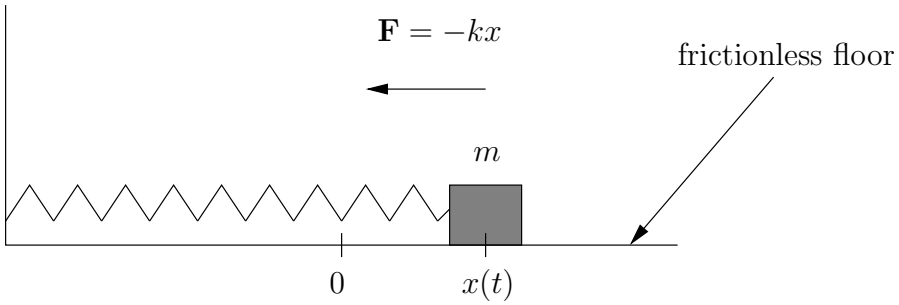
$$\frac{dv}{dt} = -g - \frac{k}{m}v. \tag{67}$$

Note that the solution of the DE is a function $v(t)$ which, when you take its derivative, yields a multiple of the functions, along with the constant $-g$. One might suspect that an exponential function makes up at least part of the solution. We'll return to this problem a little later.

We now arrive at an important point regarding the differential equations (65) and (66) - we'll focus on Eq. (66) for the velocity since it is only first-order. The point is: **We cannot solve for the velocity function $v(t)$ in Eq. (66) by simply taking antiderivatives of both sides!** This is because v on the right-hand side is a function of t – since we don't know it, we don't know what its antiderivative is! When there is no resistance, i.e., $k = 0$, the right-hand side is simply $-g$, a constant. Now we can take antiderivatives of both sides, as we did earlier for the first projectile problem. But the appearance of the unknown function $v(t)$ on the RHS – and the unknown function $x(t)$ on the RHS of Eq. (65) – makes it impossible to simply take antiderivatives of both sides to solve for the unknown function. We cannot avoid the necessity to solve differential equations involving these functions. These DEs are not very difficult to solve and, as pointed out earlier, you'll learn how to solve them in your first course on DEs, e.g., AMATH 250/251 or MATH 228.

Example 2: A mass-spring system

We now consider a "mass-spring system" as sketched below – it is a kind of generic oscillatory system. The mass m sits on a frictionless floor and is connected to a wall by means of a spring. Presumably, if you place the mass sufficiently far to the right (or left) and then release it, it will oscillate back and forth.



Oscillatory mass-spring system

We let the floor define an x-coordinate system. Furthermore, we let the origin of this system, $x = 0$, be the point at which the spring is at its natural "unstretched" state, so that it does not exert a force on the mass. If the mass is placed at $x = 0$ at rest, it will not move. As such, $x = 0$ is called the **equilibrium point** of the system. Otherwise, for $x \neq 0$, we assume that the spring exerts a restorative force, which we shall denote as $\mathbf{F}_{rest}(x) = f_{rest}(x)\,\mathbf{i}$, on the mass:

1. For $x > 0$, the spring is **extended** from its natural length. Then $f_{rest} < 0$, implying that the restorative force will act leftward, pulling the mass toward the origin.

2. For $x < 0$, the spring is **compressed** from its natural length. Then $f_{rest} > 0$, implying that the restorative force will act rightward, pushing the mass toward the origin.

The simplest model for such a restorative force, which is quite good for small displacements from equilibrium, is the linear restorative force,

$$f_{rest} = -kx, \quad k \ \text{ constant.} \tag{68}$$

You are no doubt familiar with this model – it is commonly known as **Hooke's "law"**. But it is **not** a law – it is an **approximation**. In fact, it is a **linear approximation**, much in the spirit of the linear approximation employed in the pendulum equation.

For this linear restorative force, Newton's equation $f = ma$ becomes

$$ma = -kx, \tag{69}$$

or

$$m\frac{d^2x}{dt^2} = -kx. \tag{70}$$

This is a second-order DE in $x(t)$. We can't solve this equation in the way we solved the DEs for projectile motion, i.e., by simply antidifferentiating.

Let's rewrite Eq. (70) slightly, as follows,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{71}$$

This DE is telling us that the second derivative $x''(t)$ of $x(t)$ is a negative multiple of $x(t)$. We know that sine and cosine functions exhibit this behaviour: The derivative of $\sin(at)$ is $a\cos(at)$. Taking another derivative (and keeping the constant a) gives $-a^2\sin(at)$. This shows that $a^2 = \dfrac{k}{m}$ in the above equation. It is customary to use the Greek letter ω ("omega") instead of the constant a, i.e., $\omega^2 = \dfrac{k}{m}$. And you showed in an earlier assignment that any linear combination of appropriate sine and cosine functions is also a solution to Eq. (71). The final result is as follows: All solutions to Eq. (71) have the form

$$x(t) = C_1\cos\omega t + C_2\sin\omega t, \qquad \text{where } \omega^2 = \frac{k}{m}. \tag{72}$$

Once again, these solutions are oscillatory, with frequency ω and period $T = \dfrac{2\pi}{\omega}$. The fact that they are oscillatory makes sense physically. (Recall that we assumed the floor to be frictionless.)

Equation (71) has a special name: it is called the **harmonic oscillator** equation.

The "potential energy" function

In Examples 1 (projectile problem) and 2 (mass-spring problem), but **not** Example 1(a) (projectile with air resistance), the forces acting on the mass were functions only of the position x and not time t or velocity v, i.e.,

1. Example 1: $f_g = -mg$ (a constant, actually independent of x, but that's fine),

2. Example 2: $f_{rest} = -kx$.

In such cases, as you may have seen in your Physics course, we may define an associated **potential energy** function, $U(x)$, as follows,

$$U(x) = -\int_{x_0}^{x} f(s)\, ds, \tag{73}$$

where x_0 is a conveniently chosedn reference point. Note that

$$U(x_0) = -\int_{x_0}^{x_0} f(s)\, ds = 0, \tag{74}$$

i.e., the potential function is zero at the reference point.

Some of you may not be familiar with the above definition in terms of a "definite integral." Indeed, you may not have seen definite integrals before. That's OK - before the end of this course, we'll be covering definite integrals, and all of this will make sense. We don't really need the above definition at this time. For the moment, suffice it to say that the definition in Eq. (73) implies that the potential energy $U(x)$ is the **negative of the work done by the force f in moving the mass from x_0 to x.**

A consequence of Eq. (73) is that

$$U'(x) = -f(x). \tag{75}$$

Once again, we shall understand this result a little later in this course – it is a consequence of the so-called **Fundamental Theorem of Calculus (FTC) I**. At this time, we'll simply rewrite the above equation as follows,

$$f(x) = -U'(x), \tag{76}$$

and state the following working definition:

> **Definition:** A (one-dimensional) force f that can be expressed in the form
>
> $$f(x) = -U'(x), \tag{77}$$
>
> for some function $U(x)$ is called a **conservative force**. The function $U(x)$ is known as the **potential energy function** associated with the force f.

Of course, the obvious question is, "Why is the force f called a **conservative** force?" As you know, "something" is conserved. This is the subject of the next lecture.