Lecture 18

Final comments on the theory of second-order linear ODEs

General form of nonhomogeneous second-order linear DE

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

Its associated complementary equation, the homogeneous DE

$$L[y] = y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

Recall that if y_1 and y_2 are linearly independent solutions to Eq.ⁿ (2) on an interval I, they form a "fundamental set" of solutions to Eq. (2). This means that they generate all solutions to (2) in the following sense: For any $t_0 \in I$ and any pair of initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0' \qquad (3)$$

there exists a unique pair of coefficients C_1 and C_2 such that

$$y(t) = C_1 y_1(t) + C_2 y_2(t) \qquad (4)$$

is the solution to Eq. (2) that satisfies the ICs in (3).

But we now consider the nonhomogeneous DE in (1) and make the following claim:

If $y_p(t)$ is a solution to Eq. (1) (which you might find by some method - more on this later) and y_1 and y_2 form a fundamental set of solutions to Eq. (2) (not Eq (1)) then

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \qquad (5)$$

is the general solution to Eq. (1). This means that it can be used to generate all solutions to Eq. (1).

Why is this so? For the following reason:

Suppose that $Y_1(t)$ and $Y_2(t)$ are two different solutions of Eq. (1). Then

$$\begin{aligned} L[Y_1 - Y_2] &= L[Y_1] - L[Y_2] \\ &= g(t) - g(t) \\ &= 0 \end{aligned} \qquad (6)$$

where L is the differential operator defined in (1) or (2).

But this means that $Y_1 - Y_2$ is a solution to the homogeneous DE in (2). But this means that

$$Y_1(t) - Y_2(t) = A_1 y_1(t) + A_2 y_2(t) \qquad (7)$$

for some pair of constants A_1 + A_2 (since $y_1(t)$ and $y_2(t)$ comprise a fundamental set of solutions for Eq. (2))

In other words, given ANY two solutions $Y_1(t)$ and $Y_2(t)$ of the nonhomogeneous DE in (1), they differ by some solution to the homogeneous DE. (This is somewhat analogous to ANTIDERIVATIVES:

If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, ie $F'(x) = f(x)$ and $G'(x) = g(x)$, then there exists a constant C such that $F(x) = G(x) + C$, or

$$F(x) - G(x) = C$$

In our DEs case, the constant C is replaced by a linear combination of solutions to the homogeneous DE.)

This implies that we need only to come up with one solution $y_p(x)$ of the nonhomogeneous DE in (1) and then form the general solution in Eq. (5)

Let us now state a fundamental result for nonhomogeneous (and homogeneous DE): Suppose that in Eq. (1)

$$y'' + p(t)y' + q(t)y = g(t). \qquad (1)$$

$p(t)$, $q(t)$ and $g(t)$ are continuous on an interval $I \subseteq \mathbb{R}$ ("$\subseteq$" means "is a subset or is equal to"). This means that I could be a finite interval (a,b), or the half-infinite interval, $[a,\infty)$, or the entire real line $\mathbb{R} = (-\infty,\infty)$.

Then for any $t_0 \in I$, and any pair of values $y_0, y_0' \in \mathbb{R}$, there exists a unique solution $y = \phi(x)$ to the DE in (1) which satisfies the initial conditions

$$y(t_0) = y_0 \,, \quad y'(t_0) = y_0' \qquad (8)$$

In terms of our general solution in Eq. (5), ie

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t) \qquad (5)$$

There <u>exists</u> a <u>unique</u> <u>pair</u> of values C_1 and C_2 for which Eq. (5) is a solution to Eq. (1) which satisfies the conditions in (8).

Just to summarize that we can, in principle, obtain the general solution to Eq. (1):

1. Homogeneous DE in Eq. (2)

- Constant coefficients, ie $p(t) = p$ $q(t) = q$

  Straightforward: Assume a solution of form $y = e^{rt}$

  ... & satisfies characteristic equation $r^2 + pr + q = 0$

  ....

- Non-constant coefficients - the subject of the next section

  - We'll be able to find at least one solution $y_1(t)$ and perhaps two.

  - If we can find only one solution $y_1(t)$, we can find a second, linearly independent solution $y_2(t)$ using the "Reduction of Order" method, ie

    Assume $y_2(t) = u(t)\, y_1(t)$ and find $u(t)$.

2. Nonhomogeneous DE in (1)

(a) If the function $g(t)$ in (1) is relatively simple, e.g. exponential functions, polynomials, then we can try to find a particular solution $y_p(t)$ using the "Method of Undetermined Coefficients"

(6) In more complicated situations, we can try to use the fundamental set of solutions y_1 & y_2 of the homogeneous DE to construct y_p using the "Variation of Parameters" method, ie

$$y_p = u(t)\,y_1(t) + v(t)\,y_2(t), \ \ldots$$

SERIES SOLUTIONS OF DES

** NOTE ** we now consider "x" as the independent variable instead of "t"

In many applications, the DES we encounter do not have constant coefficients. In the second order linear DE case

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

p(x) and q(x) are often polynomials. In fact, it is often the case that we encounter DEs of the form

$$a(x)\,y'' + b(x)\,y' + c(x)\,y = g(x),$$

where a(x), b(x) and c(x) are polynomials.

Examples:

$$y'' - 2xy' + y = 0$$

$$x^2 y'' + xy' + (x^2 - p^2)\,y = 0 \quad \text{(Bessel's equation)}$$

In such cases, a series solution approach may work: We assume that the solution y(x) to the DE admits a series expansion of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \dots$$

It is necessary to review some basics about infinite series which you saw in 1B Calculus.

<u>Infinite series</u> has the form

$$b_0 + b_1 + b_2 + \cdots \;=\; \text{"}\sum_{n=0}^{\infty} b_n\text{"} \qquad (1)$$

The series is said to be convergent if its partial sums converge, ie

$$\lim_{N\to\infty} S_N = L \qquad (2)$$

where

$$S_N = \sum_{n=0}^{N} b_n \qquad (N\text{th partial sum}) \qquad (3)$$

You may recall that there were several tests that can establish if a series converges or not. The one <u>absolutely</u> essential condition is that

$$\lim_{n\to\infty} b_n = 0\,, \qquad (4)$$

ie the b_n's must get smaller & smaller. In fact they have to go to zero faster than a certain rate, but we'll skip this.

Another test was the Ratio Test:

$$\text{If} \quad \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L < 1, \qquad (5)$$

then the series in (1) converges. This is based on a comparison of the $\sum b_n$ series with the standard GEOMETRIC SERIES,

$$a + ar + ar^2 + ar^3 + \dots + ar^n + \dots \qquad (6)$$

which converges for $|r| < 1$.

(Infinite) Power Series have the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \qquad (7)$$

where the $a_n \in \mathbb{R}$, $n = 0, 1, 2, \dots$. The above series is an expansion about the point $x = 0$. In general, a series expansion about the point $x_0 \in \mathbb{R}$ has the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots \qquad (8)$$

For the moment, our discussion will focus on expansions about $x_0 = 0$ - the extension to nonzero x_0 will be straightforward.

Given the series in (2), we'll need to ask for what $x \in \mathbb{R}$ does it converge, i.e. given the partial sums

$$S_N(x) = \sum_{n=0}^{N} a_n x^n$$

for what $x \in \mathbb{R}$ does

$$\lim_{N \to \infty} S_N(x) \text{ exist and is finite?}$$

As you may recall, we treat each term in (2) as a term b_n in the infinite series in (1), i.e. let

$$b_n = a_n x^n \qquad n = 0, 1, 2, \ldots$$

and use the Ratio Test. We ask: "For which $x \in \mathbb{R}$ does

$$\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = L < 1 \text{ ?"}$$

Note that

$$\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} x \right|$$

$$= |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

We want to find $x \in \mathbb{R}$ such that

$$|x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

This, of course, will depend on the coefficients a_n that make up the series of interest.

Note that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < \infty$

then by the Ratio Test, the series $\sum a_n x^n$ converges for $x \in \mathbb{R}$ such that

$$|x| L < 1 \Rightarrow |x| < \frac{1}{L} = R$$

Thus, the radius of convergence R of the series is given by

$$R = \frac{1}{L} = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

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Lecture 19 Review of Power Series (cont'd)

Examples to illustrate radius of convergence

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ $a_n = 1$

This is a geometric series $a + ar + ar^2 +$ here $a = 1$, $r = x$

Converges for $|r| = |x| < 1$

Actually the "sum" of this series is $\frac{1}{1-x}$ $|x| < 1$

2. $\sum_{n=0}^{\infty} \frac{1}{n+1} x^n$ $a_n = \frac{1}{n+1}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}} = 1$$

Series converges for $|x| < 1$

3. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = \infty$

Series converges for all $x \in \mathbb{R}$

The "sum" of the series is e^x $x \in \mathbb{R}$

If a series $\sum a_n x^n$ converges for $|x|<R$, it defines a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Furthermore,

$$f'(x) = \underbrace{a_1 + 2a_2 x + 3a_3 x^2 + \cdots \; n x^{n-1} + \cdots}$$

This series converges for $|x|<R$

$\Rightarrow$ termwise differentiation is permitted

$$f''(x) = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3 a_4 x^2 + \cdots \; n(n-1) a_n x^{n-2} \cdots$$

Also note: $f(0) = a_0$

$f'(0) = a_1$

$f''(0) = 2a_2$

$\vdots$

$f^{(n)}(0) = n!\, a_n$

$$a_n = \frac{1}{n!} f^{(n)}(0)$$

The series defining $f(x)$ then becomes

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 + \cdots$$

Taylor series of $f(x)$ at $x=0$. (or "Maclaurin series")

Partial sums are the Taylor polynomials to $f(x)$

$$S_N(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \cdots \frac{1}{N!} f^{(N)}(0)x^N = P_{N,0}(x)$$

Nth degree Taylor polynomial to $f(x)$

Finally: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $\quad g(x) = \sum_{n=0}^{\infty} b_n x^n$

$$cf(x) = \sum_{n=0}^{\infty} c a_n x^n$$

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

This allows us to manipulate, combine, etc, series termwise

We now attempt to solve some DEs by assuming series (therefore Taylor series) expansions for the solution $y(x)$

Example 1 $\qquad y'' + y = 0 \qquad$ (a) $\qquad$ (We would like to be able to generate the known solutions to this DE, ie
$y_1(x) = \cos x$
$y_2(x) = \sin x$

Step 1: Assume a solution of the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

it will be more convenient to employ Σ-notation

$$y''(x) = 2a_2 + 6a_3 x + \dots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute needed expressions into DE (a)

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad (b)$$

We need to collect terms in like powers of x^n

In first summation - let $m = n-2 \Rightarrow n = m+2$

$n=2 \Rightarrow m=0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$$

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

change m back to n

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0 \qquad (c)$$

Stop here: If you don't like this, go back to expansions

$$y'' + y = 0$$

$$2a_2 + 6a_3 x + 12a_4 x^2 + \dots \quad + a_0 + a_1 x + a_2 x^2 + \dots \qquad (d)$$

Collect like powers of x

$$\underbrace{(2a_2 + a_0)}_{\text{Corresponds to } n=0 \text{ in (c)}} + \underbrace{(6a_3 + a_1)}_{n=1 \text{ in (c)}}x + \underbrace{(12a_4 + a_2)}_{n=2 \text{ in (c)}}x^2 + \dots = 0 \qquad (e)$$

Since RHS is zero for all x in some interval, each coefficient of x^n must be zero:

$$2a_2 + a_0 = 0$$
$$6a_3 + a_1 = 0 \qquad (f)$$
$$12a_4 + a_2 = 0$$
$$\vdots$$

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \text{from (c)} \qquad (g)$$

Eq$^{\underline{n}}$ (g) is a recurrence relation:

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n \qquad n = 0, 1, 2, \ldots$$

From $a_0 \to a_2 \to a_4$
$a_1 \to a_3 \to a_5$ } In this case, not always.

$n=0 \quad a_2 = -\frac{1}{2} a_0 \qquad n=1 \quad a_3 = -\frac{1}{6} a_1$

$n=2 \quad a_4 = -\frac{1}{12} a_2 = \frac{1}{24} a_0 \qquad n=3 \quad a_5 = -\frac{1}{20} a_3 = \frac{1}{120} a_1$

$\vdots$

Above are consistent with f

Pattern: $\quad a_{2n} = (-1)^n \frac{1}{(2n)!} a_0 \qquad a_{2n+1} = (-1)^n \frac{1}{(2n+1)!} a_1$

So our series for $y(x)$ becomes

$$a_0 + a_1 x + a_2 x^2 + \ldots = a_0 \Big[\overbrace{1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots}^{\text{Taylor series expansion for } \cos x} \Big]$$

$$+ a_1 \Big[\underbrace{x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots}_{\sin x} \Big]$$

$$= a_0 \cos x + a_1 \sin x$$

In general, the series may not/will not correspond to

a standard function, ie sin, cos, exp..

Lecture 20

In the last lecture we simply stated that the following series solutions to the DE

$$y'' + y = 0,$$

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Correspond to the functions $\cos x$ and $\sin x$, respectively.

$y_1(x)$ is the Taylor series expansion of $\cos x$ at $x=0$

$y_2(x)$ " " " " " of $\sin x$ at $x=0$

Let's show this for $y_1(x)$:

Recall that Taylor series expansion for $f(x)$ at $x=0$:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f^{(2)}(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{1}{n!}f^{(n)}(0)$$

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f^{(3)}(x) = \sin x \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$a_0 = 1 \quad a_1 = 0 \quad a_2 = -\frac{1}{2!} \quad a_3 = 0 \quad a_4 = \frac{1}{4!} \quad \ldots$

We obtain the series $y_1(x) \Rightarrow y_1(x) = \cos x$.

This implies that for a fixed $x \in \mathbb{R}$, the partial sums

$$S_N(x) = \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(0) x^n$$

converge to $f(x)$.

As x increases, we need more and more terms, i.e. higher N, for $S_N(x)$ to approximate $f(x) = \cos x$ to a prescribed accuracy. This is shown in the figure on the next page. (It was taken from the textbook of Boyce & DiPrima)

Also on the next page are shown some Taylor polynomial approximations $S_N(x)$ to $\sin x$.

20.3

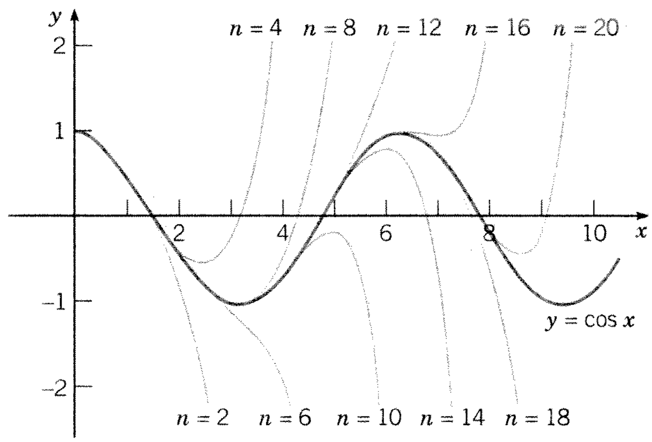


FIGURE 5.2.1 Polynomial approximations to $\cos x$. The value of n is the degree of the approximating polynomial.

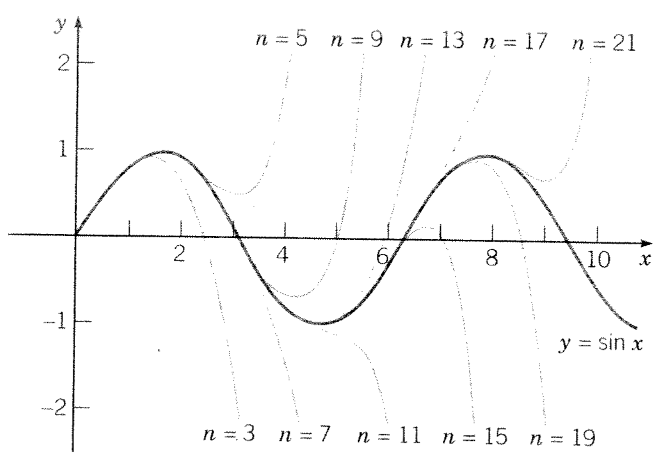


FIGURE 5.2.2 Polynomial approximations to $\sin x$. The value of n is the degree of the approximating polynomial.

Example 2 $\quad y'' - 2xy' + y = 0$

Once again, assume a series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

you can even write $\sum_{n=0} n a_n x^{n-1}$

$n=0$ makes this term zero

$$y''(x) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2}$$

Sub into DE

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}} + \sum_{n=0}^{\infty} a_n x^n$$

$m = n-2 \Rightarrow n = m+2$

$n=2, m=0$

$$\sum_{m=0} (m+2)(m+1) a_{m+2} x^m - 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$(-2n+1) a_n = -(2n-1) a_n$

$$\sum_{n=0} \Big[\underbrace{(n+2)(n+1) a_{n+2} - (2n-1) a_n}_{0 \text{ for } n=0,1,2,\ldots} \Big] x^n = 0$$

$$a_{n+2} = \frac{(2n-1)}{(n+2)(n+1)} a_n \qquad n=0,1,2,\ldots$$

Once again $\quad a_0 \to a_2 \to a_4$

$a_1 \to a_3 \to a_5$

$n=0 \quad a_2 = -\dfrac{1}{2\cdot 1} a_0$

$n=2 \quad a_4 = \dfrac{3}{4\cdot 3} a_2 = -\dfrac{3}{4\cdot 3\cdot 2\cdot 1} a_0 = -\dfrac{3}{4!} a_0$

$n=4 \quad a_6 = \frac{7}{6.5} a_4 = -\frac{3.7}{6.5.4!} a_0 = -\frac{3.7}{6!} a_0$

$n=6 \quad a_8 = \frac{11}{8.7} a_6 = -\frac{3.7.11}{8!} a_0$

a_1

$\downarrow$

$n=1 \quad a_3 = \frac{1}{3.2} a_1 = \frac{1}{3!} a_1$

$n=3 \quad a_5 = \frac{1.5}{5.4} a_3 = \frac{1.5}{5.4.3!} a_1 = \frac{1.5}{5!} a_1$

$n=5 \quad a_7 = \frac{9}{7.6} a_5 = \frac{1.5.9}{7!} a_1$

$n=7 \quad a_9 = \frac{13}{9.8} a_7 = \frac{1.5.9.13}{9!} a_1$

Result $\quad y(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$$= a_0 \left[1 - \frac{1}{2!} x^2 - \frac{1.3}{4!} x^4 - \frac{3.7}{6!} x^6 - \frac{3.7.11}{8!} x^8 - \ldots \right]$$

$$+ a_1 \left[x + \frac{1}{3!} x^3 + \frac{1.5}{5!} x^5 + \frac{1.5.9}{7!} x^7 + \frac{1.5.9.13}{9!} x^9 + \ldots \right]$$

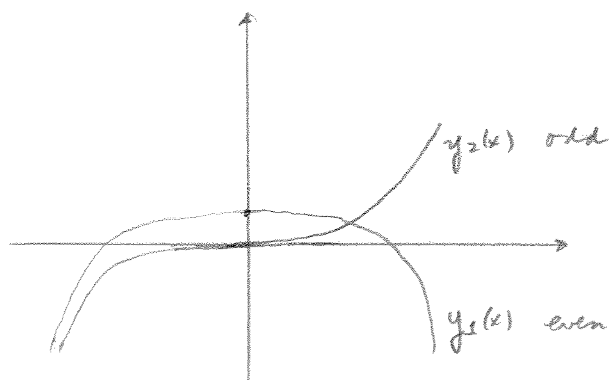
$$= a_0 y_1(x) + a_2 y_2(x)$$

$y_1(x)$ even powers of x $\Rightarrow$ even function $\quad y(-x) = y(x)$

$y_2(x)$ odd powers of x $\Rightarrow$ odd function $\quad y(-x) = -y(x)$

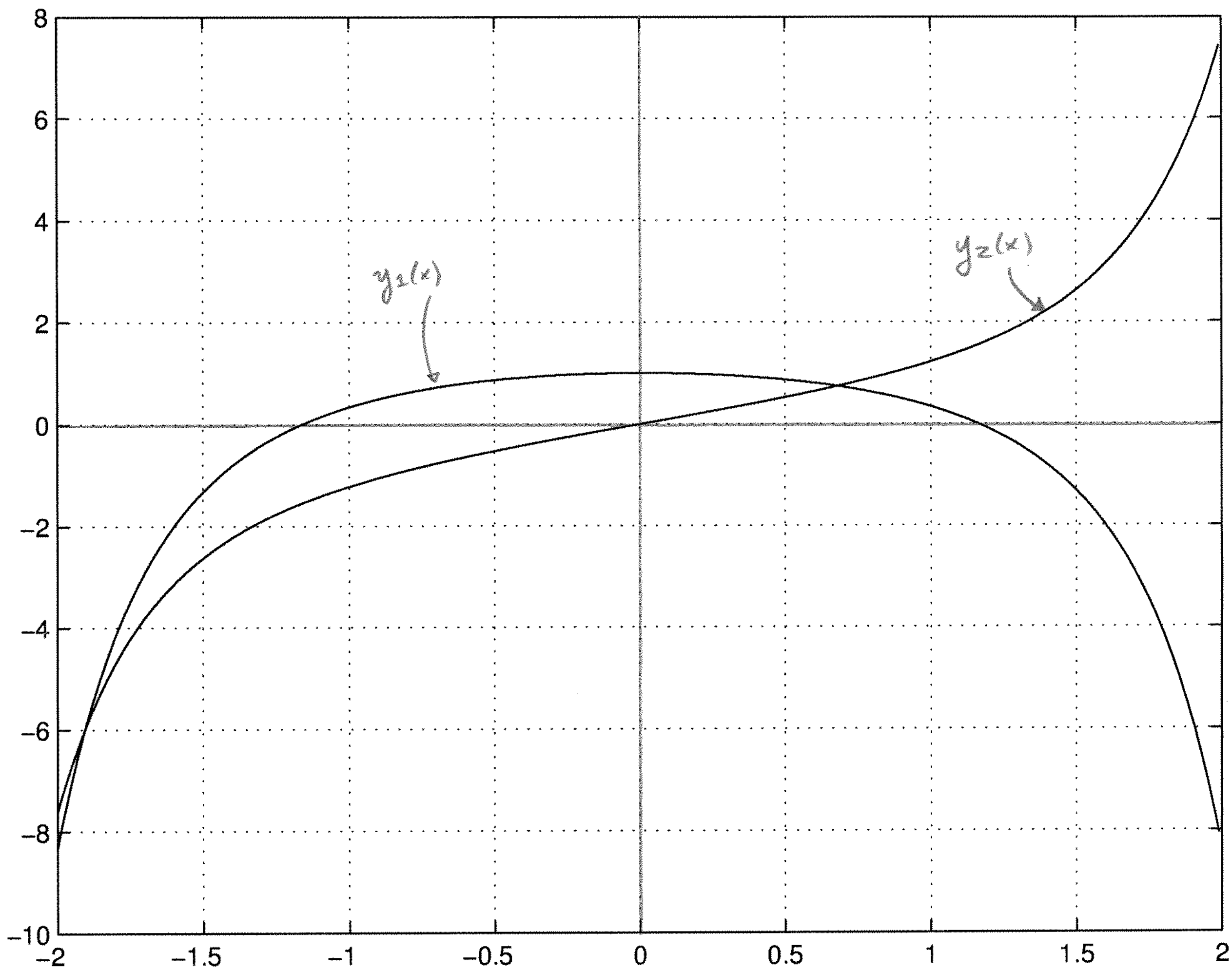
We can actually use the series to compute $y_1(x)$, $y_2(x)$, with one technicality: we should determine the interval of convergence $|x| < R$ of these series. More on this later.

Rough sketch of graphs of $y_1(x)$ and $y_2(x)$



See next page for a computer generated plot.

20.7



Solutions to $y'' - 2xy' + y = 0$ obtained by series method

The series for $y_1(x)$ and $y_2(x)$ were computed to x^{20} at the values of x from -2 to 2 spaced $\Delta x = 0.01$ apart.

Example 3 "Airy's equation" important in quantum mechanics, electromagnetism

$$y'' - xy = 0$$

Note: We didn't have time to finish this example in class, but I'm including the entire analysis as an additional example for you to see.
ERW.

Assume solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Compute derivatives

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$xy = x \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substitute into DE:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

take out n=0 term

$$2 \cdot 1 \cdot a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$a_2 = 0$

Recurrence relation $\quad a_{n+2} = \dfrac{1}{(n+2)(n+1)} a_{n-1} \qquad n=1,2,\ldots$

$\downarrow n \to n+1$

$$a_{n+3} = \frac{1}{(n+3)(n+2)} a_n \qquad n=0,1,2,\ldots$$

$a_0 \to a_3 \to a_6 \to a_9 \to \ldots$

$a_1 \to a_4 \to a_7 \to a_{10} \to$

$0 = a_2 \to a_5 \to a_8 \to a_{11} \to$ } all of these are zero

Starting at a_0

$n=0 \qquad a_3 = \dfrac{1}{3\cdot 2} a_0$

$n=3 \qquad a_6 = \dfrac{1}{6\cdot 5} a_3 = \dfrac{1}{(6\cdot 5)(3\cdot 2)} a_0$

$n=6 \qquad a_9 = \dfrac{1}{9\cdot 8} a_6 = \dfrac{1}{(9\cdot 8)(6\cdot 5)(3\cdot 2)} a_0$

Starting at a_1

$n=1 \qquad a_4 = \dfrac{1}{4\cdot 3} a_1$

$n=4 \qquad a_7 = \dfrac{1}{7\cdot 6} a_4 = \dfrac{1}{7\cdot 6\cdot 4\cdot 3} a_1$

$n=7 \qquad a_{10} = \dfrac{1}{10\cdot 9} a_7 = \dfrac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} a_1$

Series solution yields two linearly independent solutions

$$y(x) = a_0\, y1(x) + a_1\, y2(x)$$

$$y_1(x) = 1 + \frac{x^3}{2\cdot3} + \frac{x^6}{2\cdot3\cdot5\cdot6} + \cdots \frac{x^{3n}}{2\cdot3\cdot5\cdot6\cdots(3n-1)(3n)}$$

$$y_2(x) = x + \frac{x^4}{3\cdot4} + \frac{x^7}{3\cdot4\cdot6\cdot7} + \frac{x^{3n+1}}{3\cdot4\cdot5\cdot6\cdots(3n)(3n+1)}$$

It can be shown (Ratio Test) that these series converge for all $x \in \mathbb{R}$.

Some plots of the partial sum/Taylor polynomials to these series and their convergence to the solution $y_1(x)$ and $y_2(x)$ are shown on the next page.

20.11

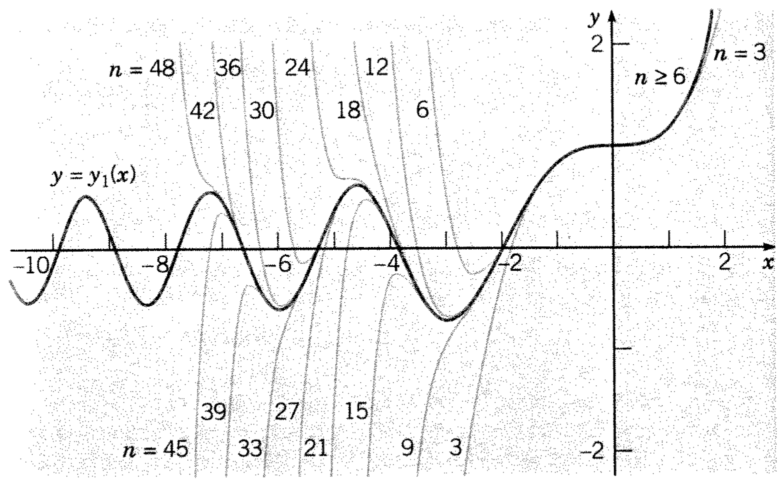


FIGURE 5.2.3 Polynomial approximations to the solution $y_1(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

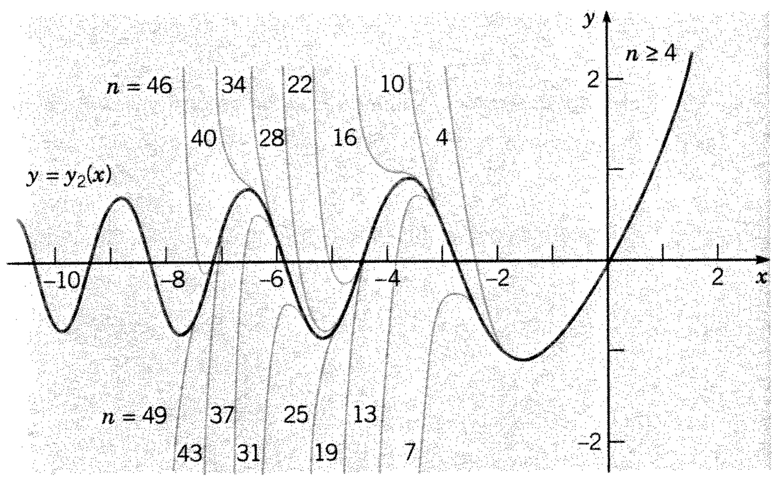


FIGURE 5.2.4 Polynomial approximations to the solution $y_2(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

the solutions y_1 and y_2 of Airy's equation are not elementary functions that you have already encountered in calculus. However, because of their importance in some physical applications, these functions have been extensively studied, and their properties are well known to applied mathematicians and scientists.

EXAMPLE 3

Find a solution of Airy's equation in powers of $x - 1$.

The point $x = 1$ is an ordinary point of Eq. (15), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$