Alternate proof of Theorem 3.3, p. 26, Course Notes

Theorem: All norms on $\mathbb{R}^n$ are equivalent.

Let $N(x) = \|x\|$ be a norm on $\mathbb{R}^n$. Show that it is equivalent to the Euclidean norm $\|x\|_2$

From Prop. 3.1, $N(x)$ is continuous on X. Consider the unit Euclidean sphere in $X = \mathbb{R}^n$, $S_1 = \{ x \in \mathbb{R}^n \mid \|x\|_2 = 1 \}$

(i.e $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$)

S_1 is closed and bounded $\Rightarrow$ compact

Therefore $N(x) = N(x_1, x_2, \ldots x_n)$ attains a maximum value B and a minimum value A. Note that $x = 0$ is not an element of $S_1 \Rightarrow A > 0$

Let $x \in \mathbb{R}^n$ and define $y = \dfrac{x}{\|x\|_2} \in S_1$

Then

$$A \leq N(y) \leq B$$

But $N(y) = N\left(\dfrac{x}{\|x\|_2}\right) = \dfrac{1}{\|x\|_2} N(x)$

$$\Rightarrow \quad A \leq \frac{1}{\|x\|_2} N(x) \leq B$$

or

$$A \|x\|_2 \leq N(x) \leq B \|x\|_2$$

Therefore $N(x) = \|x\|$ is equivalent to Euclidean norm $\|x\|_2$