AMATH 731: Applied Functional Analysis Fall 2017

Sobolev spaces, weak solutions, Part II

(To accompany Section 4.6 of the AMATH 731 Course Notes)

In the previous handout, we considered the following problem

$$u''(x) + g(x)u(x) = f(x), \qquad u(a) = u(b) = 0, \tag{1}$$

as a motivation for the study of weak solutions of DEs. Both sides of this equation were multiplied by a *test function* $\phi \in C_c^\infty(a,b)$, with compact support in (a,b):

$$\int_a^b u''(x)\phi(x)\,dx + \int_a^b g(x)u(x)\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx. \tag{2}$$

Integration by parts then yielded the following equation

$$-\int_a^b u'(x)\phi'(x)\,dx + \int_a^b g(x)u(x)\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx, \tag{3}$$

where the boundary terms disappear due to the fact that $\phi(a) = \phi(b) = 0$. Eq. (3) has the form

$$B(u,\phi) = \langle f, \phi\rangle, \tag{4}$$

where $\langle , \rangle$ denotes the usual inner product on $L^2[a,b]$ and $B(\cdot,\cdot)$ denotes a *bilinear form*, i.e., a (bounded) functional that is linear in each of its two arguments.

The goal is to determine *weak solutions* of Eq. (1) in terms of this equation involving functionals. As will be seen below, we shall write that, under appropriate conditions on B there exists a unique function u that satisfies the equation

$$B(u,v) = \langle f, v\rangle \tag{5}$$

for all $v \in H$, where H is an appropriate Hilbert space. This function u will be called the *weak solution* of Eq. (1). The so-called *Lax-Milgram Theorem* for bounded bilinear functionals, to be discussed below, will guarantee the existence of such a unique solution u.

Riesz Representation Theorem

It is first instructive to recall this fundamental result for bounded linear functionals (cf. Course Notes, p. 65).

Theorem 1 *Let $F : H \to \mathbf{R}$ be a bounded linear functional on a Hilbert space H. Then there exists a unique $z \in H$ so that $F(x) = \langle x, z\rangle$ for all $x \in H$. Moreover, $\|F\| = \|z\|$.*

Bilinear forms and the Lax-Milgram Theorem

Definition 1 *A* bilinear form *or* functional *B on a Hilbert space H is a mapping $B : H \times H \to R$ such that $a(x,y)$ is linear in each of $x, y \in H$, i.e., for all $u_1, u_2, w \in H$, and $c_1, c_2 \in \mathbf{R}$,*

$$\begin{aligned} B(c_1u_1 + c_2u_2, w) &= c_1B(u_1,w) + c_2B(u_2,w), \\ B(w, c_1u_1 + c_2u_2) &= c_1B(w,u_1) + c_2B(w,u_2). \end{aligned} \tag{6}$$

Theorem 2 *(Lax-Milgram) Let $B : H \times H \to \mathbf{R}$ be a bilinear functional such that the following conditions are satisfied: There exist constants $a, b > 0$ such that for all $u, v \in H$,*

$$|B(u,v)| \leq b\|u\|\|v\|, \tag{7}$$

$$B(u,u) \geq a\|u\|^2. \tag{8}$$

Finally, let $F : H \to \mathbf{R}$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$B(u,v) = F(v), \qquad \textit{for all } v \in H. \tag{9}$$

Proof: 1. For each fixed element $u \in H$, the mapping $v \to B(u,v)$ is a bounded linear functional on H – the boundedness follows from (7). From the Riesz Representation Theorem, there exists a unique element $w \in H$ satisfying

$$B(u,v) = \langle w, v\rangle, \qquad v \in H. \tag{10}$$

This defines a mapping $A : u \to w$ so that we shall write the above as

$$B(u,v) = \langle Au, v\rangle, \qquad u, v \in H. \tag{11}$$

(*Sneak Preview/Spoiler Alert:* Now go back to Eq. (9) and use the Riesz Representation once again to establish that

$$B(u,v) = F(v) = \langle p, v\rangle \tag{12}$$

for a unique $p \in H$. From (11) and (12) we have that

$$\langle Au, v\rangle = \langle p, v\rangle \quad \Longrightarrow \quad Au = p\,. \tag{13}$$

Ideally, our solution is then given by

$$u = A^{-1}p\,. \tag{14}$$

But does this solution exist, i.e., does A^{-1} exist?)

2. We claim that $A : H \to H$ is a bounded linear operator: For $c_1, c_2 \in R$ and $u_1, u_2 \in H$,

$$\begin{aligned}
\langle A(c_1u_1 + c_2u_2), v\rangle &= B(c_1u_1 + c_2u_2, v) \quad \text{(from 11)} \\
&= c_1B(u_1, v) + c_2B(u_2, v) \\
&= c_1\langle Au_1, v\rangle + c_2\langle Au_2, v\rangle \quad \text{(from 11)} \\
&= \langle c_1Au_1 + c_2Au_2, v\rangle.
\end{aligned} \tag{15}$$

In addition,

$$\|Au\|^2 = \langle Au, Au\rangle = B(u, Au) \leq b\|u\|\|Au\|, \tag{16}$$

implying that

$$\|Au\| \leq b\|u\|, \qquad \text{for all } u \in H. \tag{17}$$

Therefore, A is bounded. Also note that

$$a\|u\|^2 \leq |B(u,u)| \;=\; |\langle Au, u\rangle| \leq \|Au\|\|u\| \tag{18}$$

implying that

$$a\|u\| \leq \|Au\|, \qquad \text{for all } u \in H. \tag{19}$$

Therefore A is bounded from below.

3. We now show that the range of A, $R(A)$ is a closed subspace of H. Let $\{v_n\} \in R(A)$ be a Cauchy sequence with limit $v \in H$. Since each $v_n \in R(A)$, there exists a $u_n \in H$ such that $Au_n = v_n$. From (19),

$$a\|u_n - u_m\| \leq \|Au_n - Au_m\| = \|v_n - v_m\|, \tag{20}$$

implying that the sequence $\{u_n\}$ is also Cauchy. Let $u \in H$ denote the limit of this sequence. Since A is bounded, hence continuous, it follows that

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} Au_n = A(\lim_{n\to\infty} u_n) = Au. \tag{21}$$

Therefore $v \in R(A)$, implying that $R(A)$ is closed.

4. From (19), $A : H \to R(A)$ is bounded from below, implying that an inverse operator A^{-1} exists, i.e., A is one-to-one.

5. We now show that $R(A) = H$. Assume that $R(A) \neq H$. Then, since $R(A)$ is closed, there exists a nonzero element $w \in H$ such that $w \in R(A)^{\perp}$. But $Aw \in R(A)$. This implies that $a\|w\|^2 \leq B(w, w) = \langle Aw, w\rangle = 0$, contradicting (8). Therefore $R(A) = H$.

6. Now consider the given functional $F(v)$. From the Riesz Representation Theorem, there exists a $p \in H$ such that $F(v) = \langle p, v\rangle$. From 4. and 5., there exists a $u \in H$ such that $Au = p$, i.e., $u = A^{-1}p$, so that, from (11),

$$B(u, v) = \langle p, v\rangle = F(v). \tag{22}$$

7. Finally, we show that there is at most one element $u \in H$ satisfying (9). Assume the contrary, i.e., that $B(u, v) = F(v)$ and $B(\bar{u}, v) = F(v)$ for all $v \in H$. Then, by linearity, $B(u - \bar{u}, v) = 0$ for $v \in H$. Set $v = u - \bar{u}$ to give, from (8), $a\|u - \bar{u}\|^2 \leq B(u - \bar{u}, u - \bar{u}) = 0$. Therefore $u = \bar{u}$, and the theorem is proved.

Application to a one-dimensional boundary-value problem

We first consider the following boundary value problem:

$$-u''(x) = f(x), \quad 0 \leq x \leq 1, \quad u(0) = u(1) = 0. \tag{23}$$

A physical interpretation of this problem is that $u(x)$ is the *transverse* deflection (in the y-direction) of a homogeneous string at point x under the influence of a force $f(x)$ (actually a force density – more on this later) acting in the y-direction. (We also assume the use of appropriate units that simplify the form of the problem.) Note that in this problem, the string is clamped at both ends, i.e., $x = 0$ and $x = 1$.

Brief note on the origin of the above equation: Recall the 1D PDE for a homogeneous vibrating string with an external force $f(x)$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x)\,, \quad 0 \leq x \leq 1\,, \tag{24}$$

where $c^2 = T/\rho_0$. (T is the tension in the string and ρ_0 the lineal mass density (mass/unit length), both assumed to be constant.) For simplicity, we have set $c = 1$. Eq. (23) corresponds to the steady-state, or time-independent, solution $u(x, t) = u(x)$, i.e., $\dfrac{\partial u}{\partial t} = 0$.

Using arguments similar to those in the previous handout, the total strain or elastic energy of the string is

$$\mathcal{U} = \frac{1}{2}\int_0^1 [u'(x)]^2\, dx, \tag{25}$$

and the total work of the external force is

$$\mathcal{W} = \int_0^1 f(x)u(x)\, dx. \tag{26}$$

The associated energy functional $J(u)$ for this problem is given by

$$J(u) = \frac{1}{2}\int_0^1 [u'(x)]^2\, dx - \int_0^1 f(x)u(x)\, dx, \tag{27}$$

These three equations have the same form as for the one-dimensional rod problem examined in the previous handout. Following the same type of variational method as before, one can show that the minimizer u_0 of $J(u)$ corresponds to the solution of (23):

$$J(u_0 + \epsilon v) = J(u_0) + \epsilon DJ(u_0)v + \frac{1}{2}\epsilon^2 \int_0^1 [v'(x)]^2\, dx\,. \tag{28}$$

But $DJ(u_0) = 0$ since u_0 is a solution to Eq. (23), implying that

$$J(u_0 + \epsilon v) > J(u_0)\,. \tag{29}$$

Classical treatment and its limitations

In the classical picture – indeed, the one employed earlier in this course – one assumes that $f \in C[0,1]$ so that $u \in C^2[0,1]$. The solution u of (23) may be expressed in terms of f using the Green's function associated with this boundary-value problem:

$$u(x) = \int_0^1 g(x,y)f(y)\, dy, \tag{30}$$

where

$$g(x,y) = \begin{cases} y(1-x), & 0 \le y \le x \le 1, \\ x(1-y), & 0 \le x \le y \le 1, \end{cases} \tag{31}$$

Since f is assumed to be $C[0,1]$, everything is nice here, and u is twice differentiable.

But what if f is "not so nice"? Well, if f were an L^1 function, the integral would still make sense and $u(x)$ would be defined. But what about $u'(x)$?

For example, what about the case where $f(x)$ is a force concentrated at a point? Such forces are often modelled with a Dirac "delta function", which corresponds to a force of finite strength, say A, (with $A > 0$ corresponding to an upward-pointing force, $A < 0$ to a downward-pointing force) concentrated at a point $0 < a < 1$, implying that the force density is infinite. Bypassing rigour for the moment, such a force would be written as

$$f(x) = A\delta(x-a). \tag{32}$$

Even in classical treatments, this expression for $f(x)$ is usually inserted into into (30), keeping in mind the property that, for any $v \in C[0,1]$,

$$\int_0^1 v(x)\delta(x-a)\, dx = v(a). \tag{33}$$

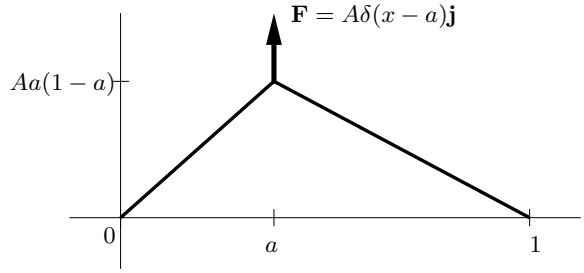
The result is

$$u(x) = A\int_0^1 g(x,y)\delta(y-a)\, dy = Ag(x,a), \tag{34}$$

implying that

$$u(x) = \begin{cases} Ax(1-a), & 0 \le x \le a \le 1, \\ Aa(1-x), & 0 \le a \le x \le 1, \end{cases} \tag{35}$$

Thus the graph of $u(x)$ has a triangular shape, with a vertex at $x = a$, for which $u(a) = Aa(1-a)$:



The result, as is well known in undergraduate courses, is that $u(x)$ is not differentiable at $x = a$; it is, however, piecewise differentiable. In other words, we have to move away from the "very nice" space $u \in C^2[0,1]$. From the discussion of the previous handout, we see that this solution for $u(x)$ is well accomodated in the "energy space", E_R, or, equivalently, the Sobolev space $W^{1,2}$, since

$$\int_0^1 [u'(x)]^2 \, dx < \infty. \tag{36}$$

Finally, we make a comment regarding the relationship of the Dirac "delta function" $\delta(x)$ and the Green's function $g(x,y)$ for this problem – a relationship that applies to other similar boundary-value problems. For simplicity, let the amplitude of the point force be $A = 1$. Then, for a point force $f(x)$ located at $a \in (a,b)$, i.e., $f(x) = \delta(x-a)$, the function $u(x) = g(x,a)$ – the Green's function itself – is seen to be the solution to the equation

$$-u'' = \delta(x-a). \tag{37}$$

This result also applies to the problem of "point electric charges" and associated potentials. It is often stated in textbooks and can be proved rigorously using generalized functions and associated weak derivatives – see E. Zeidler, *Applied Functional Analysis, Applications to Mathematical Physics*, Springer-Verlag (1997), p. 158-164.

Weak derivative/Sobolev space treatment

We now relax the restriction that $f \in C[0,1]$ in (23) to $f \in L^2[0,1]$ and apply the method of weak derivatives in Sobolev spaces to this problem. We'll work in the space $W_c^{1,2}(0,1)$, the space of functions with compact support on $(0,1)$ and norm

$$\|u\|_{1,2} = \left(\int_0^1 ([u(x)]^2 + [u'(x)]^2) \, dx \right)^{1/2} \tag{38}$$

The most important aspect of this space is that the (weak) derivative $u'(x)$ is L^2-integrable, cf. Eq. (36).

Note: In the literature, the following notation, $W_c^{k,p}(D)$ is often denoted as $W_0^{k,p}(D)$. In addition, the $p = 2$ Sobolev spaces are often denoted as

$$H^k(D) = W^{k,2}(D), \tag{39}$$

acknowledging that these spaces are Hilbert spaces. Once again, the superscript "0", i.e., $H_0^k(D)$, will be used to denote the subspace of functions with compact support on $(0,1)$. This is also the notation used in the AMATH 731 Course Notes. We adopt this notation below.

We now return to Eq. (23),

$$-u''(x) = f(x), \quad 0 \le x \le 1, \quad u(0) = u(1) = 0\,, \tag{40}$$

multiply both sides of it with an arbitrary element $v \in H_0^1(D)$ and integrate by parts to give

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \quad v \in H_0^1(D), \tag{41}$$

where it is understood that $u'(x)$ represents the generalized or weak derivative of $u(x)$. Eq. (41) has the form

$$B(u,v) = F(v), \tag{42}$$

where the bilinear functional $B(u,v)$ and the linear functional $F(v)$ are defined, respectively, as

$$\begin{aligned} B(u,v) &= \langle u', v'\rangle = \int_0^1 u'(x)v'(x)\,dx, \\ F(v) &= \langle f, v\rangle = \int_0^1 f(x)v(x)\,dx. \end{aligned} \tag{43}$$

We now seek to apply the Lax-Milgram theorem to establish the existence of a unique solution $u \in H_0^1(D)$ to (41), hence to (23).

First of all, the linear functional $F(v)$ is bounded:

$$|F(v)| = |\int_0^1 f(x)v(x)\,dx| \le \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2} \left(\int_0^1 |v(x)|^2\,dx\right)^{1/2} < \infty. \tag{44}$$

As for the bilinear functional $B(u,v)$, it is bounded from above:

$$\begin{aligned} |B(u,v)| &\le \int_0^1 |u'(x)||v'(x)|\,dx \\ &\le \left(\int_0^1 |u'(x)|^2\,dx\right)^{1/2} \left(\int_0^1 |v'(x)|^2\,dx\right)^{1/2} \\ &\le \|u\|_{H_0^1}\|\|v\|_{H_0^1}, \end{aligned} \tag{45}$$

thereby establishing that the condition (7) for the Lax-Milgram Theorem is satisfied. Establishing the second condition (8), i.e., bounding $B(u,u)$ from below, is a little trickier. We have to resort to Poincaré's Inequality, cf. Theorem 4.9, AMATH 731 Course Notes, p. 68. Actually, the one-dimensional version of Corollary 4.2, on p. 69, is sufficient for this problem: There exists a constant $c_1 > 0$ such that

$$\|u\|_{H_0^1}^2 \le (c_1+1)\int_0^1 |u'(x)|^2\,dx, \quad \text{for all } u \in H_0^1(0,1). \tag{46}$$

(In the 1D case, the proof is quite simple, see below.) Since

$$B(u,u) = \int_0^1 [u'(x)]^2\,dx, \tag{47}$$

it follows that

$$\frac{1}{c_1+1}\|u\|_{H_0^1}^2 \;\le\; B(u,u). \tag{48}$$

Therefore, B satisfies both conditions of the Lax-Milgram Theorem. We can conclude that there exists a unique element $u \in H_0^1(0,1)$ that satisfies (42), therefore (41), and therefore the boundary-value problem (23) in the generalized sense.

Proof of Eq. (46): We first consider $u \in C_0^1(0,1)$. From the Fundamental Theorem of Calculus, for $x \in (0,1)$:

$$\int_0^x u(t)u'(t)\,dt = \frac{1}{2}u(x)^2 - \frac{1}{2}u(0)^2 = \frac{1}{2}u(x)^2. \tag{49}$$

But from the Cauchy-Schwarz inequality,

$$\int_0^1 |u(x)u'(x)|\,dx \le \left[\int_0^1 u(x)^2\,dx\right]^{1/2}\left[\int_0^1 u'(x)^2\,dx\right]^{1/2}. \tag{50}$$

Combining these two results, we have that

$$u(x)^2 \le 2\left[\int_0^1 u(x)^2\,dx\right]^{1/2}\left[\int_0^1 u'(x)^2\,dx\right]^{1/2}. \tag{51}$$

This implies that

$$\int_0^1 u(x)^2\,dx \le 2\left[\int_0^1 u(x)^2\,dx\right]^{1/2}\left[\int_0^1 u'(x)^2\,dx\right]^{1/2}. \tag{52}$$

Squaring both sides and rearranging yields

$$\int_0^1 u(x)^2\,dx \le 4\int_0^1 [u'(x)]^2\,dx. \tag{53}$$

Adding $\int_0^1 u'(x)^2\,dx$ to both sides shows that Eq. (46) is satisfied for any $u \in C_0^1(0,1)$, with $c_1 = 4$.

For the case $u \in H_0^1(0,1)$, there is a sequence $\{u_n\} \subset C_0^1(0,1)$ such that $u_n \to u$ in H_0^1-norm. Since Eq. (46) is satisfied by all $u_n \in C_0^1(0,1)$, it will be satisfied for u.

Finite elements and the Ritz method

We now outline a procedure, based on the so-called *method of finite moments*, and the Ritz method, to provide approximate solutions to the boundary-value problem (23). In what follows, we outline the application of the method to a boundary-value problem over the general interval $[a,b]$, i.e., $u(a) = u(b) = 0$. Clearly, for the above problem, $a = 0$, $b = 1$.

First, divide the interval $[a,b]$ into $n+1$ equal subintervals using the partition

$$a_0 = a < a_1 < a_2 < \cdots < a_n < a_{n+1} = b, \tag{54}$$

where

$$a_j = \frac{j}{n+1}(b-a). \tag{55}$$

By definition, a *finite element*

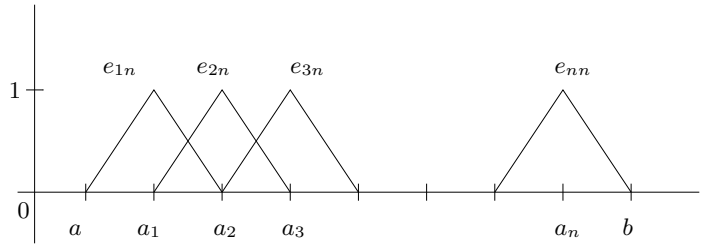
$$e_{in} : [a,b] \to \mathbf{R}, \qquad i = 1, 2, \cdots, n, \tag{56}$$

is a piecewise (triangular-shaped) linear function with

$$e_{in}(a_i) = 1, \quad \text{and} \quad e_{in}(a_j) = 0, \quad \text{for all } j \ne i. \tag{57}$$

We define

$$X_n = \text{span}\{e_{1n}, e_{2n}, \cdots, e_{nn}\}. \tag{58}$$



Then $u_n \in X_n$ if

$$u_n(x) = \sum_{i=1}^{n} c_{in} e_{in}. \tag{59}$$

Note: Each function e_{in} satisfies the boundary condition $e_{in}(a) = e_{in}(b) = 0$, which implies that

$$u_n(a) = u_n(b) = 0, \qquad \text{for all } u_n \in X_n. \tag{60}$$

The function $u_n \in X_n$ is piecewise linear and $u_n(a_i) = c_{in}$ for all $i = 1, 2, \cdots, n$. Therefore the space X_n consists of all piecewise linear functions with respect to the points $a, a_1, a_2, \cdots, a_n, b$ which satisfy the boundary-value condition $u(a) = u(b) = 0$.

We now return to the energy functional $J(u)$ associated with this BVP, cf. Eq. (27), but now in the Sobolev space $H_0^1(a, b)$:

$$J(u) = \frac{1}{2} \int_a^b [u'(x)]^2 \, dx - \int_0^1 f(x) u(x) \, dx, \quad u \in H_0^1(a, b). \tag{61}$$

The minimization of this functional with respect to functions $u_n \in X_n$ is a *Ritz problem*:

$$\min_{u_n \in X_n} F(u_n). \tag{62}$$

This represents a minimization problem with respect to the real variables $c_{1n}, c_{2n}, \cdots, c_{nn}$ in Eq. (59). If u_n is a solution to (62), then

$$\frac{\partial}{\partial c_{jn}} F(u_n) = 0, \qquad j = 1, 2, \cdots, n. \tag{63}$$

This produces the so-called *Ritz equations*:

$$\int_a^b u_n' e_{jn}' \, dx = \int_a^b e_{jn} f \, dx, \qquad u_n \in X_n, \quad j = 1, 2, \cdots, n. \tag{64}$$

Explicitly, we have a linear system of equations in the unknowns c_{in}:

$$\sum_{i=1}^{n} c_{in} \int_a^b e_{in}' e_{jn}' \, dx = \int_a^b e_{jn} f \, dx, \quad j = 1, 2, \cdots, n. \tag{65}$$

For each n, this linear system has the form

$$\mathbf{A}\mathbf{c} = \mathbf{f}, \tag{66}$$

where

$$a_{ij} = \langle e_i', e_j' \rangle, \quad c_i = c_{in}, \quad f_i = \langle e_{jn}, f \rangle, \qquad 1 \le i, j \le n. \tag{67}$$

The matrix $\mathbf{A}$ is quite concentrated near the diagonal, given that the finite element e_{jn} overlaps only with itself and its two immediate neighbours $e_{j\pm1,n}$.

Proposition 1 (The Ritz method via finite elements) *Let $f \in C[a,b]$. Then the above Ritz method converges to the unique solution u of the boundary-value problem (23) in the sense of the Sobolev space $H_0^1(a,b) = W_0^{1,2}(a,b)$, i.e.*

$$\|u - u_n\|_{1,2} \to 0 \quad as \quad n \to \infty. \tag{68}$$

For a proof of this proposition, as well as a rigorous estimate of the error, see E. Zeidler, *Applied Functional Analysis, Applications to Mathematical Physics*, Springer-Verlag (1997).

Weak solutions of PDEs

Second-order elliptic PDEs

Here we consider briefly boundary-value problems of the form

$$\begin{aligned} Lu &= f \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D, \end{aligned} \tag{69}$$

where D is an open, bounded set of $\mathbf{R}^n$. Here, $f : D \to \mathbf{R}$ and $g : \partial D \to \mathbf{R}$ is given. L denotes a second-order partial differential operator, expressed in so-called *divergence form*,

$$Lu = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} u) + \sum_{i=1}^{n} b_i(x)\partial_{x_i} u + c(x)u, \tag{70}$$

which is amenable for treatments involving integration by parts, e.g., energy methods, weak solutions. The requirement that $u = 0$ on the boundary ∂D is known as the *Dirichlet boundary condition.*

Another class of problems is as follows,

$$\begin{aligned} Lu &= 0 \quad \text{in } D, \\ u &= g \quad \text{on } \partial D. \end{aligned} \tag{71}$$

This *generalized Dirichlet problem* in $\mathbf{R}^2$, for which $f = 0$, is the subject of Question No. 6 in Problem Set 5 of the AMATH 751 Course Notes. Such a boundary-value problem would arise when trying to find the electrostatic potential $u(x)$ inside a charge-free region D, produced by given charge density g on the boundary ∂D.

With an eye to applications, we shall assume that

$$a_{ij} = a_{ji}, \quad 1 \le i, j \le n. \tag{72}$$

As well, we consider only *elliptic* partial differential operators L, i.e., those for which the following condition holds: There exists a constant $C > 0$ such that

$$\sum_{i=1}^{n} a_{ij}(x)\xi_i\xi_j \;\ge\; C|\xi|^2, \tag{73}$$

for (almost) all $x \in D$ and all $\xi \in \mathbf{R}^n$. For L to be elliptic means that the symmetric $n \times n$ matrix $A(x)$ is positive definite, with smallest eigenvalue $\lambda \ge C$.

Special case: $a_{ii} = 1$, $a_{ij} = 0$ for $i \ne j$, $b_i = 0$, $c = 0$ in (70), in which case $L = -\nabla^2 = -\triangle$, the negative Laplacian operator.

Physical interpretation: Second-order elliptic PDEs are generalizations of Laplace's and Poisson's equations. Let us first review briefly some applications that give rise to Laplace's equation. Typically,

a function u will denote the amount or density of some quantity (e.g., temperature, electrostatic potential, chemical concentration) in equilibrium. Then if V is an arbitrary subregion within D, with smooth boundary ∂V, then the net flux of u through ∂V is zero, i.e.

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = 0, \tag{74}$$

where $\mathbf{F}$ denotes the *flux density* (discussed below) and $\mathbf{n}$ the unit outer normal vector field to ∂S. From the Divergence Theorem,

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = \int_V \text{div } \mathbf{F} \, dx = 0, \tag{75}$$

implying that

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = 0 \quad \text{ in } D, \tag{76}$$

since V was arbitrary. In many applications, it is a reasonable assumption that the flux $\mathbf{F}$ is proportional to the gradient ∇u, but pointing in the opposite direction, since the flow will be from regions of higher concentration to those of lower concentration. In other words,

$$\mathbf{F} = -K \nabla u, \qquad K > 0. \tag{77}$$

Substitution into (76) yields *Laplace's equation*

$$\nabla \cdot (\nabla u) = \nabla^2 u = 0. \tag{78}$$

Some examples:

u	flux law in Eq. (77)
chemical concentration	Fick's "law" of diffusion
temperature	Fourier's "law" of heat conduction
electrostatic potential	Ohm's "law" of electrical conduction

A classical example in electrostatics comes from the fundamental equation,

$$\text{div } \mathbf{E}(x) = -\frac{\rho(x)}{\epsilon_0}, \tag{79}$$

where $\mathbf{E}(x)$ denotes the electric field at $x \in \mathbf{R}^n$ due to charge density $\rho(x)$. (Here, ϵ_0 is the *permittivity of the vacuum.*) Since

$$\mathbf{E} = -\nabla V, \tag{80}$$

where V denotes the associated electrostatic potential function, we have

$$\nabla^2 V = \frac{\rho}{\epsilon_0}, \tag{81}$$

or *Poisson's equation.* Of course, in the absence of charge, this equation reduces to Laplace's equation.

In the more general case, i.e., the operator in Eq. (70), the second-order terms involving the a_{ij} represents *diffusion* within region D – the coefficients a_{ij} describe the anisotropic, heterogeneous nature of the medium. The first-order terms involving the b_i represent *transport* within D. The zeroth-order term cu describes local *creation* or *depletion* (for example, in chemical applications, due to reactions that either produce or consume the chemical).

In what follows, it will be assumed that

$$a_{ij}(x),\ b_i(x),\ c(x) \in C(D). \tag{82}$$

This assumption could be relaxed even further to $L^\infty(D)$. Furthermore, we assume that

$$f \in L^2(D). \tag{83}$$

We work in the Hilbert space of functions $H_0^1(D) = W_0^{1,2}(D)$. We now multiply both sides of the equation $Lu = f$ by a function $v \in H_0^1(D)$ and integrate the first term by parts to obtain

$$\int_D \left[\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b_i u_{x_i} v + cuv \right] dx = \int_D fv\, dx, \tag{84}$$

where

$$u_{x_i} = \partial_{x_i} u, \quad etc.. \tag{85}$$

Eq. (84) now has the form

$$B(u,v) = F(v), \quad \text{for } u, v \in H_0^1(D), \tag{86}$$

where

$$F(v) = \langle f, v \rangle = \int_D fv\, dx \tag{87}$$

is the linear functional and

$$B(u,v) = \int_D \left[\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b_i u_{x_i} v + cuv \right] dx \tag{88}$$

is the bilinear form associated with the divergence form elliptic operator L defined in (70).

We now say that $u \in H_0^1(D)$ is a *weak solution* of the boundary-value problem (69) if

$$B(u,v) = F(v) \tag{89}$$

for all $v \in H_0^1(D)$.

Theorem 3 *(Energy estimates) There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B(u,v)| \leq \alpha \|u\|_{H_0^1(D)} \|v\|_{H_0^1(D)} \tag{90}$$

$$\beta \|u\|_{H_0^1(D)} \leq B(u,u) + \gamma \|u\|_{L^2(D)}. \tag{91}$$

Proof: See *Partial Differential Equations*, by L.C. Evans, AMS (1998), pp. 300-301.

Note that if $\gamma > 0$ in the above estimates, then $B(,)$ does not precisely satisfy the second hypothesis of the Lax-Milgram theorem. A slight "tinkering" must be performed – see the book by Adams, pp. 301-302. In the case of the Laplacian operator, the above theorem (Energy estimates) holds true for $\gamma = 0$.

Second-order parabolic PDEs

We simply mention briefly that the idea of weak solutions may be applied PDEs that involve time, often referred to as PDE evolution equations. Second-order parabolic PDEs are generalizations of the heat equation, involving a first-order time derivative. (This is in contrast to *hyperbolic equations* that involve second-order time derivatives, e.g., the wave equation.) In fact, the weak solution approach provides the basis of the so-called *Galerkin's method* of computing approximations to these equations.

In what follows, we assume D to once again be an open, bounded subset of $\mathbf{R}^n$ and define $D_T = D \times (0, T]$ for some fixed time $T > 0$. We now consider the initial/boundary-value problem

$$\begin{aligned} u_t + Lu &= f \quad \text{in } D_T, \\ u &= 0 \quad \text{on } \partial D \times [0, T], \\ u &= g \quad \text{on } \partial D \times \{t = 0\}. \end{aligned} \tag{92}$$

L denotes for each time t a second-order partial differential operator in divergence form,

$$Lu = -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x,t)\partial_{x_i} u) + \sum_{i=1}^{n} b_i(x,t)\partial_{x_i} u + c(x,t)u, \tag{93}$$

We also assume that the operator L is uniformely elliptic for each time $t \in [0, T]$, i.e., there exists a constant $C > 0$ such that (cf. Eq. (73),

$$\sum_{i=1}^{n} a_{ij}(x,t)\xi_i\xi_j \;\geq\; C|\xi|^2. \tag{94}$$

In this case, one says that the operator $\dfrac{\partial}{\partial t} + L$ is (uniformly) *parabolic*.

Special case: Once again, $a_{ii} = 1$, $a_{ij} = 0$ for $i \neq j$, $b_i = 0$, $c = 0$ in (70), in which case $L = -\nabla^2 = -\triangle$, the negative Laplacian operator, so that the PDE

$$\frac{\partial u}{\partial t} - \nabla^2 = f \tag{95}$$

becomes the heat equation, with source term f.

In physical applications, general second-order parabolic equations describe the time-evolution of a quantity, e.g., chemical concentration, within a region D.

Proceeding in a manner quite similar to that for elliptic equations, we assume that

$$\begin{aligned} &a_{ij}(x),\ b_i(x),\ c(x) \in C(D), \\ &f \in L^2(D_T), \\ &g \in L^2(D). \end{aligned} \tag{96}$$

We also assume that $a_{ij} = a_{ji}$.

As for the elliptic case, we work in the Hilbert space of functions $H_0^1(D) = W_0^{1,2}(D)$. Multiplying the operator in (93) with a $v \in H_0^1(D)$ and integrating the first term by parts yields a time-dependent bilinear form

$$B(u, v; t) = \int_D \left[\sum_{i,j=1}^{n} a_{ij}(\cdot,t)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b_i(\cdot,t)u_{x_i}v + c(\cdot,t)uv \right] dx, \qquad u, v \in H_0^1(D). \tag{97}$$

It is then tempting to write down an equation involving time-varying bilinear and linear functionals, B and F, respectively, involving the function $u = u(x,t)$. The standard procedure, however, is to associate with u a mapping

$$\mathbf{u} : [0,T] \to H_0^1(D) \tag{98}$$

defined by

$$[\mathbf{u}(t)](x) := u(x,t) \quad (x \in D, 0 \le t \le T). \tag{99}$$

In other words, u will not be considered as a function of x and t but rather as a mapping $\mathbf{u}$ of t into the space $H_0^1(D)$ of functions of x.

Similarily, one defines

$$\mathbf{f} : [0,T] \to L^2(D) \tag{100}$$

by

$$[\mathbf{f}(t)](x) := f(x,t) \quad (x \in D, 0 \le t \le T). \tag{101}$$

We now multiply the PDE $u_t + Lu = f$ by a fixed function $v \in H_0^1(D)$ and integrate by parts to produce

$$\langle \mathbf{u}', v \rangle + B(u,v) = \langle \mathbf{f}, v \rangle, \tag{102}$$

for each $0 \le t \le T$, where the prime represents differentiation with respect to time.

Definition 2 *We say that a function*

$$\mathbf{u} \in L^2(0,T; H_0^1(D) \tag{103}$$

is a weak solution *of the parabolic initial/boundary-value problem (92) provided that*

$$\langle \mathbf{u}', v \rangle + B(u,v) = \langle \mathbf{f}, v \rangle, \tag{104}$$

for each $v \in H_0^1(D)$ and a.e. time $0 \le t \le T$, and

$$\mathbf{u}(0) = g. \tag{105}$$

(There is another technical point regarding the domain of definition of $\mathbf{u}'$ but we omit it here – see Adams, p. 352.)