# AMATH 731: Applied Functional Analysis Fall 2009

# Weak convergence in a Hilbert space and the Ritz approximation method

(To accompany Section 4.8 of the AMATH 731 Course Notes)

## Some introductory remarks on the "Rayleigh-Ritz method" of approximation

We first recall the definition of a positive operator in a Hilbert space, as given in the previous set of notes:

**Definition 1** *A bounded linear self-adjoint operator $L$ on a Hilbert space $H$ is said to be* **positive** *if $\langle Lx, x\rangle \geq 0$ for all $x \in H$. This is often denoted as "$T \geq 0$" or "$0 \leq T$". $L$ is said to be* **strictly positive** *if $\langle Lx, x\rangle > 0$ for all $x \neq 0$. This is often denoted as "$T > 0$" or "$0 < T$".*

And recall that a simple consequence of this definition is the following:

**Theorem 1** *Let $L$ be a strictly positive, compact, self-adjoint operator on an infinite dimensional Hilbert space $H$. Then the eigenvalues of $L$ are positive, with $\lambda_1 = \| L \|$ and*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots > 0. \tag{1}$$

*Furthermore* $\lim_{n\to\infty} \lambda_n = 0$.

Strictly positive operators are often encountered in applications, e.g., vibrational modes and associated energies of a mechanical system (e.g., a vibrating rod or plate), Hamiltonian operators for quantum mechanical systems (the eigenvalues correspond to energies). In these cases, the Hilbert space $H$ of concern is infinite dimensional and therefore difficult to work with practically. A goal of many studies is to estimate the eigenvalues and eigenvectors of such systems. A natural starting-point is to consider finite-dimensional representations, or "truncations" of the infinite-dimensional problem, with the hope that as the dimension $N$ of the finite-dimensional representation is increased, better estimates of the "true" energies are obtained. This is the essence of the so-called "Rayleigh-Ritz" methods. The usual formulation of such methods is in terms of a variational problem involving the minimization of a functional, e.g., energy. The following discussion is intended to give a very rough idea of the philosophy of this approach. A slighly more detailed description is provided in the later sections.

One may consider the truncations of our infinite-dimensional system as being the result of working with a finite collection of of linearly independent basis functions, say, $\{v_1, v_2, \cdots v_N\}$. In many cases, the minimization of the functional of interest – call it $J$ – involves finding the optimal linear combination of these basis functions, i.e.

$$c_1 v_1 + c_2 v_2 + \cdots + c_N v_N. \tag{2}$$

(This is the essence of the *variational method* of quantum chemistry, where one tries to estimate energies of atomic and molecular systems.) We now seek to minimize the functional $J(c_1, c_2, \cdots, c_N)$ in terms of the coefficients $c_k$. One often obtains a system of $N$ linear equations of the form

$$A^{(N)} c^{(N)} = \lambda^{(N)} c^{(N)}, \tag{3}$$

in other words, an eigenvalue-eigenvector relation. The eigenvalues $\lambda_n^{(N)}$ of $A^{(N)}$ will be estimates of, for example, the energy of the system. As $N$ is increased, it is hoped that the estimates $\lambda_n^{(N)}$ approach (and even converge to?) the exact values.

**Note:** In the book by Naylor and Sell, p. 661, the Rayleigh-Ritz method is discussed in terms of finding the eigenvalues of a positive, compact, self-adjoint operator $T : H \to H$. The technique outlined by the authors is essentially the procedure of extracting eigenvalues and eigenvectors of a compact self-adjoint operator $L$ in the Course Notes starting with Proposition 4.9 (p. 79) and its proof. Due to the positivity, the first eigenvalue $\mu_1 = \| T \|$ will be determined by

$$\mu_1 = \sup_{\langle x,x\rangle = 1} \langle Tx, x\rangle. \tag{4}$$

Now let $e_1$ be an eigenvector of $T$ associated with $\mu_1$. As in the course notes, let $H_1 = \{x \in H : \langle x, e_1 = 0\}$. then

$$\mu_2 = \sup_{\langle x,x\rangle=1, x\in H_1} \langle Tx, x\rangle, \tag{5}$$

and so on. The only problem – and the authors point this out – is that the above method requires a knowledge of the eigenvectors $e_1, e_2, \cdots$. It does not provide a method for finding these eigenvectors!

Here is where the truncation approach outlined earlier will come into play. We simply work with a finite dimensional representation of $T$ which is provided by a finite collection of suitable linearly independent functions $v_1, v_2, \cdots, v_N$ in $H$. The matrix $A^N$ in Eq. (3) will simply be the matrix representation of $T$ in the $v_k$ basis, with elements

$$a_{ij} = \langle Av_i, v_j\rangle, \quad 1 \le i, j \le N. \tag{6}$$

The eigenvalues $\lambda_k^N$, $k = 1, 2, \cdots, N$, of $A^N$ will be approximations to the true eigenvalues $\mu_k$ of $A$. Of $H$ is infinite-dimensional, these approximations will be *lower bounds*, since the optimization (i.e., finding the supremum) is performed over a smaller space, i.e., $\mathbf{R}^N$.

## Weak convergence in a Hilbert space

Recall the idea of convergence in a normed linear space $X$: We say that the sequence $\{x_n\}$ converges to a limit $x \in X$, often written compactly as "$x_n \to x$, if $\|x_n - x\| \to 0$ as $n \to \infty$. (Here, the $\epsilon - \delta$ definition is understood.) In a Hilbert space $H$, the norm is defined by means of the inner product on $H$, so that

$$\|x_n - x\| = \sqrt{\langle x_n - x, x_n - x\rangle}. \tag{7}$$

In many situations, however, it is useful to consider another kind of convergence – a weaker convergence that is defined in terms of inner products. In the case of function spaces, such a convergence – important in the study of *distributions* and *generalized derivatives* – will be defined in terms of inner products/integration with respect to smooth functions called *test functions*. This will be the subject of several future lectures in this course. For the moment, let it suffice to say that this weaker convergence is important in the study of the Ritz approximation method, which has been outlined in the earlier section. It also provides the basis for a number of approximation schemes, e.g., Galerkin, finite element, for the numerical solution of PDEs.

As a motivation, suppose that we have been able to generate a sequence of elements $\{x_n\} \in H$ that serve as approximations of an element $x \in H$. However, the best that we can do is to show that

$$\lim_{n\to\infty} \langle x_n, y\rangle \to \langle x, y\rangle \quad \text{for all } y \in H. \tag{8}$$

One may well be tempted to conclude that the $x_n$ actually converge to $x$, i.e., $\|x_n - x\| \to 0$, as $n \to \infty$. But this is not necessarily true: The Cauchy-Schwartz inequality furnishes a relation that "goes the wrong way", i.e.,

$$|\langle x_n - x, y\rangle| \le \|x_n - x\|\|y\|. \tag{9}$$

The above "convergence" of the $x_n$ is known as *weak convergence*, which we now formally define. The following discussion, which is in no way complete, is based upon the treatment found in the book, *Functional Analysis*, by Lebedev, Vorovich and Gladwell (Section 4.6).

**Definition 2** *A sequence $\{x_n\}$ in a normed linear space $X$ is said to be a* weak Cauchy sequence *if, for every continuous linear functional $F : X \to \mathbf{R}$ (or $\mathbf{C}$), the sequence $\{F(x_k)\}$ is a Cauchy sequence on $\mathbf{R}$ (or $\mathbf{C}$). The sequence $\{x_n\} \subset X$ is said to* converge weakly *to $x \in X$ if, for every continuous linear functional $F$ on $X$,*

$$\lim_{n\to\infty} F(x_n) = F(x). \tag{10}$$

Some books refer to this convergence as "weak*" ("weak-star") convergence. There are also many notations for weak convergence. Here we shall use the notation

$$x_n \xrightarrow{w^*} x. \tag{11}$$

In some books (e.g., Lebedev *et al.* normal convergence, i.e., $\|x_n - x\| \to 0$, is referred to as *strong* convergence in order to differentiate it from weak convergence. We shall refrain from using the word "strong" since it has other connotations. As perhaps suspected, normal convergence is "stronger" than weak convergence:

**Problem 1** *Let $X$ be a normed linear space. Show that if $\{x_n\} \subset X$ is a Cauchy sequence, then it is a weak Cauchy sequence. Show also that if $x_n \to x \in X$, then $x_n \stackrel{w^*}{\to} x$.*

**Problem 2** *Let $X$ be a normed linear space. Show that a sequence $\{x_n\} \subset X$ cannot have two distinct weak limits.*

Note that up to this point, weak convergence has been defined over a general normed space $X$. From this point onward, unless otherwise stated, we shall be considering weak convergence in a Hilbert space $H$ – of course, a complete normed linear space equipped with an inner product. Recall that, from the Riesz Representation theorem, *every* bounded linear functional $F : H \to \mathbf{R}$ (or $\mathbf{C}$) may be expressed in terms of an inner product. For each such $F$, there exists a $z \in H$ such that

$$F(x) = \langle x, z \rangle \quad \text{for all } x \in H. \tag{12}$$

This yields the following definition for weak convergence in a Hilbert space.

**Definition 3** *Let $H$ be a Hilbert space. A sequence $\{x_n\} \subset H$ is said to be a* weak Cauchy sequence *if, for every $y \in H$, the sequence $\{\langle x_n, y\rangle\}$ is a Cauchy sequence. The sequence $\{x_n\} \subset X$ is said to* converge weakly *to $x$ if, for every $y \in H$,*

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle x, y\rangle. \tag{13}$$

**Theorem 2** *If $\{x_n\} \subset H$, $x_n \stackrel{w^*}{\to} x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

**Proof:** Consider

$$\begin{aligned} \|x_n - x\|^2 &= \langle x_n - x, x_n - x\rangle \\ &= \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2. \end{aligned} \tag{14}$$

Since $x_n \stackrel{w^*}{\to} x$, we have

$$\langle x_n, x\rangle + \langle x, x_n\rangle \to 2\langle x, x\rangle = 2\|x\|^2. \tag{15}$$

This implies that

$$\lim_{n\to\infty} \|x_n - x\|^2 = \lim_{n\to\infty} \left[\|x_n\|^2 - \|x\|^2\right] = 0. \tag{16}$$

**Theorem 3** *In a finite-dimensional Hilbert space $H$, convergence and weak convergence are equivalent.*

The following theorem, although seemingly obvious, requires a little work. Its proof is to be found in the book of Lebedev *et al.* (Section 4.6, p. 120).

**Theorem 4** *A weak Cauchy sequence $\{x_n\}$ in a Hilbert space is bounded.*

It is important in the establishment of the next important theorem, which we state without proof. (For details, again consult Lebedev *et al.*, pp. 120-122.) Before stating this theorem, however, we need the following definition.

**Definition 4** *Let $X$ be a normed linear space. A countable system $g_1, g_2, \cdots, \subset X$ is said to be* complete *in $X$ if for any $x \in X$ and any $\epsilon > 0$, there is a finite linear combination of the $g_k$ such that*

$$\left\| x - \sum_{k=1}^{n} c_k g_k \right\| \le \epsilon. \tag{17}$$

**Note:** An orthonormal basis $\{\phi_n\}$ of a Hilbert space is complete. However, a set of linearly independent functions that are not necessarily orthogonal to each other, can also be complete. (Example: the set of powers $1, x, x^2, \cdots$ on $L^2([0,1])$.

In the orthonormal basis case, suppose that we can achieve the above inequality for a given $\epsilon$ with a set of Fourier coefficients $\{c_1, c_2, \cdots, c_{n_1}\}$. Then for a smaller value $\epsilon' < \epsilon$, we simply employ the above coefficients along with an additional set $\{c_{n_1+1}, \cdots, c_{n_2}\}$, $n_2 > n_1$. In the case of a complete system that is not orthonormal, however, the "lower order" coefficients $\{c_1, c_2, \cdots, c_{n_1}\}$ may change as we reduce $\epsilon$.

We now state the following important results.

**Theorem 5** *A sequence $\{x_n\}$ is a weak Cauchy sequence in a Hilbert space iff*

1. *$\{x_n\}$ is bounded in $H$, i.e., there is an $M$ such that $\|x_n\| \leq M$, and*
2. *for any $g_k \in H$ from a system $\{g_k\} \in H$ which is complete in $H$, the sequence $\{\langle x_n, g_k\rangle\}$, $n = 1, 2, \cdots$, is a Cauchy sequence.*

The next result is closely related:

**Theorem 6** *A sequence $\{x_n\}$ converges weakly to $x \in H$ iff*

1. *$\{x_n\}$ is bounded in $H$, and*
2. *for any $g_k \in H$ from a system $\{g_k\} \in H$ which is complete in $H$,*

$$\lim_{n\to\infty} \langle x_n, g_k\rangle = \langle x, g_k\rangle. \tag{18}$$

This is a nice result – it implies that we need only establish weak Cauchy or weak convergence of a sequence $\{x_n\} \in H$ by examining only the inner products of the $x_n$ with the countable set of elements $g_k$ in a complete system.

Since weak convergence differs from strong convergence, it is necessary to define the terms *weakly closed* and *weakly complete.*

**Definition 5** *Let $X$ be a normed linear space. A set $S \subset X$ is said to be* weakly closed *in $X$ if all of its weak limit points are in $X$. In other words, $x_n \xrightarrow{w^*} x \in X$ implies that $x \in S$.*

**Definition 6** *Let $X$ be a normed linear space. $X$ is said to be* weakly complete *if every weak Cauchy sequence (defined above) converges weakly to an element $x \in X$.*

**Theorem 7** *A Hilbert space (a complete inner product space) is weakly complete.*

**Proof:** Let $\{x_n\} \in H$ be a weak Cauchy sequence. We must show that it converges weakly to a limit $x \in H$. For any $y \in H$, we may define the linear functional

$$F(y) = \lim_{n\to\infty} \langle y, x_n\rangle. \tag{19}$$

From either of Theorems 4 or 5, it follows that the $\|x_n\|$ are bounded, i.e., $\|x_n\| \leq M$ for all $n$. This implies that

$$|F(y)| \leq M\|y\|, \quad \text{i.e. } \|F\| \leq M. \tag{20}$$

Thus, $F$ is continuous. By the Riesz Representation Theorem, there exists an $x \in H$ such that

$$F(y) = \langle y, x\rangle, \quad \text{for all } y \in H, \tag{21}$$

where $\|x\| = \|F\| \leq M$. This means that

$$\lim_{n\to\infty} \langle y, x_n\rangle = \langle y, x\rangle \quad \text{for all } y \in H, \tag{22}$$

implying that $x$ is the weak limit of the $\{x_n\}$.

**Corollary 1** *The closed ball about zero, $\overline{B_r(0)} \subset H$, $r > 0$, is weakly closed.*

Suppose $\{x_n\} \subset \overline{B_r(0)}$ and and $x_n \xrightarrow{w^*} x$. Then $\|x_n\| \leq r$. From the previous Theorem, it also follows that $\|x\| \leq M$, implying that $x \in \overline{B_r(0)}$. Thus $\overline{B_r(0)}$ is weakly closed.

Here is a most interesting result.

**Theorem 8** *Let $X$ be an inner product space. A weakly closed set $S \subset X$ is closed. However, a closed set need not be weakly closed.*

**Proof:**

1. Suppose that $S \subset X$ is weakly closed and let $\{x_n\} \subset S$ be a (strongly) convergent sequence with limit $x \in X$. We need to prove that $x \in S$. Let $F(x)$ be any continuous linear functional on $X$. Then

$$|F(x_n) - F(x)| = |F(x_n - x)| \leq \|F\| \, \|x_n - x\| \to 0, \tag{23}$$

implying that $\{x_n\}$ converges weakly to $x$. Since $S$ was assumed to be weakly closed, it follows that $x \in S$. Therefore $S$ is also closed.

2. To show that a closed set is not necessarily weakly closed, let $X = L^2([0,1])$ and $S \subset X$ be the unit ball $\|x\| = 1$. The set $S$ is closed since $x_n \to x$ and $\|x_n\| = 1$ implies $\|x\| = 1$. Now consider the functions $x_n \in S$ given by

$$x_n(t) = \sqrt{2}\sin(n\pi t). \tag{24}$$

(It is easy to check that $\langle x_n, x_n \rangle = 1$.) Given any function $f(t) \in L^2([0,1])$, the *Riemann-Lebesgue Lemma* implies that the integrals

$$\int_0^1 f(t)\sin(n\pi t) \to 0 \quad \text{as} \quad n \to \infty. \tag{25}$$

This implies that the functions $x_n$ converge weakly to 0, i.e., $x_n \overset{w^*}{\to} 0$, since

$$\langle x_n, f \rangle = \sqrt{2}\int_0^1 f(t)\sin(n\pi t)\, dx \to 0, \quad \text{for all } f \in X. \tag{26}$$

However, the function $x = 0$ is not in $S$. Therefore $S$ is not weakly closed.

A (strongly) closed *set* in an inner product space need not be weakly closed, but:

**Theorem 9** *A closed* subspace $M$ *of a Hilbert space* $H$ *is weakly closed.*

### Weak compactness

Finally, we should mention the idea of weak compactness:

**Definition 7** *Let $X$ be an inner product space. The set $S \subset X$ is said to be* weakly compact *if every sequence in $S$ contains a subsequence which converges weakly to an element $x \in S$.*

The following theorem is proved in Lebedev *et al.*, p. 124.

**Theorem 10** *Let $H$ be a Hilbert space. A set $S \subset H$ is weakly compact iff it is bounded and weakly closed.*

## Ritz approximation in a Hilbert space

This section follows the discussion provided in the book, *Functional Analysis*, by Lebedev, Vorovich and Gladwell (Section 4.8). It begins by examining the problem of finding best approximations in a Hilbert space setting. Recall that this problem was examined in a previous handout, first in terms of Banach spaces, and then in terms of projections in Hilbert spaces. The method discussed below may be applied to other, more general problems.

Let $H$ be a Hilbert space – assumed to be real, for simplicity – $M \subset H$ a closed subspace of $H$ and $x_0 \in M^\perp$. For illustrative purposes, we are interested in finding the unique point $p$ that minimizes the distance functional

$$F(x) = \| x - x_0 \|, \quad x \in M. \tag{27}$$

The Ritz method, due to Walter Ritz (1909) may be broken down into four steps, which we briefly describe below. (Before Ritz, similar ideas were employed by Lord Rayleigh.)

**Step 1.** ***Set up the approximation problem and study its solutions***

Assume that $M$ has a complete, linearly independent (i.e., not necessarily orthonormal) basis $\{g_k\}$ Let $M_n$ be the subspace spanned by $(g_1, g_2, \cdots, g_n)$. The existence of a unique $x \in M_n$ that minimizes $F(x)$ on $M_n$ is guaranteed by the Projection Theorem, Section 4.3 of the Course Notes. Let $x_n$ denote the minimizer.

This implies that the real-valued function

$$f(t) = [F(x_n + tg_m)]^2 = \| x_n - x_0 + tg_m \|^2 = \langle x_n - x_0 + tg_m, x_n - x_0 + tg_m \rangle, \tag{28}$$

of the real variable $t$ assumes a minimum value at $t = 0$ for each $g_m$, $m = 1, 2, \cdots, n$. In fact, $f(t)$ is differentiable,

$$f'(0) = \frac{d}{dt}\langle x_n - x_0 + tg_m, x_n - x_0 + tg_m \rangle|_{t=0} = 2\langle x_n - x_0, g_m \rangle = 0. \tag{29}$$

This implies that $x_n - x_0$ is orthogonal to each $g_m$, $m = 1, 2, \cdots, n$. Equivalently,

$$\langle x_n, g_m \rangle = \langle x_0, g_m \rangle, \quad m = 1, 2, \cdots, n. \tag{30}$$

If we let

$$x_n = \sum_{k=1}^{n} c_{kn} g_k, \tag{31}$$

the orthogonality requirements in (30) yield a set of $n$ simultaneous linear equations for the $c_{kn}$:

$$\sum_{k=1}^{n} c_{kn} \langle g_k, g_m \rangle = \langle x_0, g_m \rangle, \quad m = 1, 2, \cdots, n. \tag{32}$$

Since the set $\{g_1, \cdots, g_n\}$ is linearly independent, the solution to this equation is unique, which we leave as an exercise.

Note that if the set $\{g_1, g_2, \cdots, g_n\}$ were an *orthonormal set*, then the solution of the above system would be, simply,

$$c_{mn} = \langle x_0, g_m \rangle, \quad m = 1, 2, \cdots, n, \tag{33}$$

i.e., the Fourier coefficients of $x_0$ with respect to the $\{g_k\}$ basis, a result that we have already seen.

**Step 2. A priori** ***estimate of the approximation***

An *a priori* estimate is one which can be obtained without actually knowing the approximation, or even whether it exists. It can be a very coarse estimation. One starts with the definition of $x_n$:

$$\| x_n - x_0 \|^2 \leq \| x - x_0 \|^2, \quad x \in M_n. \tag{34}$$

Since $x = 0$ lies in $M_n$, we have

$$\| x_n - x_0 \| \leq \| x_0 - 0 \| = \| x_0 \| . \tag{35}$$

From the triangle inequality (what else?),

$$\| x_n \| \leq \| x_n - x_0 \| + \| x_0 \| \leq 2 \| x_0 \|, \quad n > 0. \tag{36}$$

which is the required estimate. This will be good enough, as we see below.

**Step 3.** ***Weak passage to limit*** **as** $n \to \infty$

This now requires the ideas of weak convergence discussed earlier. Recall that the set $M$, in which we are searching for the minimizer, was assumed to be a closed subspace of $H$. From Theorem 9, $M$ is weakly closed. From (36), the sequence $\{x_n\} \in H$ is bounded. From Theorem 10, it follows that the set $\{x_n\}$ is weakly compact, implying the existence of a weakly convergent subsequence $\{x_{n_k}\}$, with weak limit $x^* \in M$: For any fixed $m$, we can pass to the limit $n_k \to \infty$ in the orthogonality equation (29), i.e.,

$$\langle x_{n_k} - x_0, g_m \rangle = 0, \tag{37}$$

so that

$$\langle x^* - x_0, g_m \rangle = 0. \tag{38}$$

This is possible because $\langle x, g_m \rangle$, is a continuous linear functional.

After some additional analysis (Lebedev *et al.*, p. 127), it can be shown that $x^*$ is the solution to the original problem of minimizing $F(x)$ in Eq. (27).

**Step 4.** ***Study the convergence of the sequence of approximations*** **as** $n \to \infty$

In Step 3, weak convergence to the solution, i.e., $x_{n_k} \overset{w^*}{\to} x^*$, was shown. It is possible to show (Lebedev *et al.*, p. 127-128), that the entire sequence $\{x_n\}$ converges to $x^*$. Moreover, this result may be strengthened, and the (strong) convergence of the sequence $\{x_n\}$ to the solution may be established, in other words,

$$\lim_{n\to\infty} x_n = x^*. \tag{39}$$

**Application to the minimization of a general functional**

The procedure outlined in Steps 1-4 may also be applied to the problem of minimizing the functional

$$J(x) = \| x \|^2 + 2\Phi(x), \tag{40}$$

where $\Phi(x)$ is a continuous linear functional, in the following way. From the Riesz representation theorem,

$$\Phi(x) = \langle x, -x_0 \rangle \quad \text{for some} \quad x_0 \in H, \tag{41}$$

so that

$$J(x) = \| x \|^2 + 2\langle x, -x_0 \rangle = \| x - x_0 \|^2 - \| x_0 \|^2 . \tag{42}$$

Since $x_0$ is fixed, the problem of minimizing $J(x)$ is equivalent to that of minimizing

$$F(x) = \| x - x_0 \| \tag{43}$$

for $x \in H$. The solution to this problem is simply $x = x_0$. An immediate reaction is, "But we don't know $x_0$!", which is true, but we'll be able to eliminate $x_0$, as we now show.

As was done above, we suppose that $\{g_m\}$ is a complete system of basis elements in $H$ such that any finite set $\{g_1, g_2, \cdots, g_n\}$ is linearly independent. The $n$th Ritz approximation is once again

$$x_n = \sum_{k=1}^{n} c_{kn} g_k. \tag{44}$$

The orthogonality requirements in Eq. (30) once again yield the set of linear equations

$$\langle x_n, x_m \rangle = \sum_{k=1}^{n} c_{kn} \langle g_k, g_m \rangle = \langle x_0, g_m \rangle, \quad m = 1, 2, \cdots, n. \tag{45}$$

But from Eq. (41), we have that $\langle x_0, g_m \rangle = -\Phi(g_m)$, so that the above system becomes

$$\sum_{k=1}^{n} c_{kn} \langle g_k, g_m \rangle = -\Phi(g_m), \quad m = 1, 2, \cdots, n. \tag{46}$$

Thus, $x_0$ has been eliminated, since we have employed the original functional $\Phi$.

In summary, for each $n$, the system of linear equations in (46) has a unique solution $c_{1n}, c_{2n}, \cdots, c_{nn}$. From the analysis outlined briefly above, it follows that when $\Phi(x)$ is a continuous linear functional, the sequence $\{x_n\}$ yielded by (44) converges strongly to the unique minimizer of the quadratic functional $J(x)$.

Many problems in mechanics, in various separable Hilbert spaces, can be treated in this way, as is pointed out in the book by Lebedev *et al.*.

**A "less elegant" derivation: Bypassing the Riesz Representation Theorem**

Eq. (46) can be derived by straightforward "brute-force" minimization. If we substitute the Ritz approximation in Eq. (44) into the functional $J(x)$ in Eq. (40), we obtain

$$\begin{aligned} J(x_n) &= \left\langle \sum_{k=1}^{n} c_{kn} g_k, \sum_{l=1}^{n} c_{ln} g_l \right\rangle + 2\Phi\left( \sum_{k=1}^{n} c_{kn} g_k \right) \\ &= \sum_{k,l=1}^{n} c_{kn} c_{ln} \langle g_k, g_l \rangle + 2 \sum_{k=1}^{n} c_{kn} \Phi(g_k). \end{aligned} \tag{47}$$

We now impose the stationarity conditions,

$$\frac{\partial J}{\partial c_{mn}} = 0, \qquad m = 1, 2, \cdots, n. \tag{48}$$

Differentiating, we obtain

$$\frac{\partial J}{\partial c_{mn}} = \sum_{l=1} c_{ln} \langle g_m, g_l \rangle + \sum_{k=1}^{n} c_{kn} \langle g_k, g_l \rangle + 2\Phi(g_m) = 0. \tag{49}$$

A simple rearrangement yields Eq. (46).

## Appendix: "Energy Space" Norms

Now let us return to the fundamental result of Theorem 4.15 of the Course Notes concerning compact and self-adjoint operators on a Hilbert space $H$. We shall change the notation somewhat:

> If $A$ is a compact and self-adjoint operator on a (separable) Hilbert space $H$, then there exists a sequence of orthonormal eigenvectors $\{x_i\}$ and associated eigenvalues $\{\lambda_i\}$ such that $Ax_i = \lambda_i x_i$. Furthermore, the eigenvectors $\{x_i\}$ form an orthonormal basis for $H$, iff $\lambda = 0$ is not an eigenvalue for $A$. Moreover, for any $x \in H$,
>
> $$Ax = \sum_{k=1}^{N} \lambda_k \langle x, x_k \rangle x_k. \tag{50}$$

(Note that we have omitted the result that $\lim_{n\to\infty} \lambda_n = 0$. In this way, the above statements also apply to the case where $H$ is finite dimensional.)

Let us now consider the case that the compact, self-adjoint operator $A$ is also positive, implying that all eigenvalues are positive. In this case, it is not difficult to show that we can define a norm in terms of $A$,

$$\| x \|_A = \langle Ax, x \rangle^{1/2}, \tag{51}$$

and a corresponding inner product

$$\langle x, y \rangle_A = \langle Ax, y \rangle. \tag{52}$$

The completion of $H$ with respect to this norm is called $H_A$.

A consequence of this construction is that the sequence $\{y_k\}$ defined by

$$y_k = \frac{1}{\sqrt{\lambda_k}} x_k, \quad k = 1, 2, \cdots, \tag{53}$$

forms an orthonormal basis for $H_A$:

$$\langle y_k, y_l \rangle_A = \langle Ay_k, y_l \rangle = \lambda_k \langle y_k, y_l \rangle^{1/2} = \frac{\lambda_k}{\sqrt{\lambda_k}\sqrt{\lambda_l}} \langle x_k, x_l \rangle = \delta_{kl}, \tag{54}$$

so that $\| y_k \|_A = 1$.

**Example:** Consider the eigenvalue equation associated with the (clamped) vibrating string problem:

$$y''(t) + \mu y(t) = 0, \quad y(0) = y(\pi) = 0. \tag{55}$$

The eigenvalues are $\mu_k = k^2$, $k = 1, 2, \cdots$, with associated orthonormal eigenfunctions

$$y_k(t) = \sqrt{\frac{2}{\pi}} \sin\ kt, \qquad k = 1, 2, \cdots. \tag{56}$$

The $y_k$ are also orthonormal eigenfunctions of the compact, self-adjoint, positive integral operator associated with Eq. (55). The associated eigenvalues are given by

$$\lambda_k = \frac{1}{\mu_k} = \frac{1}{k^2}, \quad k = 1, 2, \cdots. \tag{57}$$

Note that $\lambda_k \to 0$ as $k \to \infty$.

Now let $W$ be the Sobolev space of functions $W_c^{1,2}(0,\pi)$ with inner product

$$\langle y, z\rangle_W = \int_0^\pi y'(t)\overline{z'(t)}\, dt. \tag{58}$$

$W$ is a Hilbert space. (We shall discuss this space in more detail in a future lecture. Let it suffice here to state that these functions have compact support $[0,\pi]$.) The norm $\|\cdot\|_{1,2}$ defined by this inner product is a norm for $W_c^{1,2}$.

Now rewrite the eigenvalue problem in (55) as $-y'' = \mu y$, multiply by a function $z \in W$ and integrate from 0 to $\pi$:

$$-\int_0^\pi y''(t)\overline{z(t)}\, dt = \mu \int_0^\pi y(t)\overline{z(t)}\, dt. \tag{59}$$

Integration by parts on the left produces

$$\int_0^\pi y'(t)\overline{z'(t)}\, dt = \mu \int_0^\pi y(t)\overline{z(t)}\, dt, \tag{60}$$

or

$$\langle y, z\rangle_W = \mu\langle y, z\rangle. \tag{61}$$

But we shall now rewrite this equation as

$$\langle y, z\rangle_W = \mu\langle Ay, z\rangle_W, \tag{62}$$

or, equivalently,

$$\langle Ay, z\rangle_W = \lambda\langle y, z\rangle_W, \qquad \lambda = \frac{1}{\mu}, \tag{63}$$

where the operator $A$ is defined as

$$\langle Ay, z\rangle_W = \int_0^\pi y(t)\overline{z(t)}\, dt. \tag{64}$$

In fact, Eq. (63) may be rewritten in the following way,

$$\langle Ay - \lambda y, z\rangle_W = 0 \quad \text{for all} \;\; z \in W, \tag{65}$$

in order to emphasize the eigenvalue-eigenvector relation involving the operator $A$.

**Here is an important point:** We consider $W$ to be our "normal" space of functions, on which the operator $A$, with eigenvalue $\lambda$, is defined by the above relationship. In the space $H_A$,

$$\| y \|_A^2 = \langle Ay, y\rangle_W = \int_0^\pi |y(t)|^2\, dt, \tag{66}$$

which implies that $H_A = L^2(0,\pi)$. (This is the space that we usually consider to be "normal".) The functions $y_k(t)$ listed in Eq. (56) form an orthonormal basis for $L^2(0,\pi)$, hence $H_A$.

From the discussion preceding this example, the eigenfunctions $x_k(t)$ of the "normal" space $W$ will then be given by

$$x_k = \sqrt{\lambda_k}\, y_k = \sqrt{\frac{2}{\pi}}\frac{\sin\, kt}{k}. \tag{67}$$

These functions form a basis for $W = W_c^{1,2}(0,\pi)$.