Summary of computations estimating eigenvalues of integral operators

1 <u>Simple example</u> (presented in course handout on spectrum of linear operator)

$$(Lf)(t) = \int_0^1 st\, f(s)\, ds = t \int_0^1 s f(s)\, ds$$

$Lv = \lambda v \qquad v = t \qquad \lambda = \frac{1}{3}$

We can actually compute that $\|L\| \leq \frac{1}{3}$

<u>Case 1:</u> Let $P_n(t)$ be family of orthogonal polynomials on $[0,1]$

$P_0(t) = 1$

$P_1(t) = 2\sqrt{3}\,(t - \frac{1}{2})$

$P_2(t) = 6\sqrt{5}\,(t^3 - t + \frac{1}{6})$

(obtained by Gram-Schmidt on $1, t, t^2, \ldots$ using MAPLE)

Matrix representation of L in P_n basis

$$\underline{\underline{L}} = L_{ij} = \langle LP_i, P_j \rangle \qquad \underline{\underline{L}} = \begin{bmatrix} \begin{matrix} \frac{1}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{12} \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$$

This is a consequence of fact that

$L: f \longrightarrow at$ for some $a \in \mathbb{R}$

$at \in \text{span}\{P_0, P_1\}$

Eigenvalues $\boxed{\lambda_1 = \frac{1}{3}}$ $\quad \lambda_2 = \lambda_3 = \cdots = 0.$

Eigenvector: $\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\frac{1}{4} v_1 + \frac{\sqrt{3}}{12} v_2 = \frac{1}{3} v_1$

$\Rightarrow \frac{\sqrt{3}}{12} v_2 = \frac{1}{12} v_1 \Rightarrow v_2 = \frac{1}{\sqrt{3}} v_1$

Choose $v_1 = 1 \quad v_2 = \frac{1}{\sqrt{3}}$

$\Rightarrow P_0 + \frac{1}{\sqrt{3}} P_1 = 1 + 2(t - \frac{1}{2}) = 2t$

$\Rightarrow v(t) = t$ is an eigenvector ✓

2×2 matrix $\underline{\underline{A}} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{12} \end{bmatrix}$ has eigenvalues $\lambda_1 = \frac{1}{3}$ $\lambda_2 = 0$

$(\det \underline{\underline{A}} = 0)$

What is 0-eigenvector of A?

$$\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{12} \\ \frac{\sqrt{3}}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{4} v_1 + \frac{\sqrt{3}}{12} v_2 = 0 \quad \Rightarrow \quad v_2 = -\frac{12}{\sqrt{3}} \cdot \frac{1}{4} v_1 = -\sqrt{3}\, v_1$$

Choose $v_1 = 1$ $v_2 = -\sqrt{3}$

$$\Rightarrow \quad P_0 - \sqrt{3}\, P_1 = 1 - 6\left(t - \tfrac{1}{2}\right) = 4 - 6t \quad \text{is a 0-eigenvector}$$

Check: $L(4-6t) = t \int_0^1 s(4-6s)\, ds = t \int_0^1 (4s - 6s^2)\, ds = t \left[2s^2 - 2s^3\right]_0^1$

$= 0.$

Furthermore: $\boxed{L(P_k) = 0 \qquad k \geq 2}$

$$(L P_k)(t) = t \int_0^1 s\, P_k(s)\, ds$$

But $s = \frac{1}{2\sqrt{3}} P_1(s) + \frac{1}{2} P_0(s)$

$$\Rightarrow \quad (L P_k)(t) = t \int_0^1 \left[\frac{1}{2\sqrt{3}} P_1(s) + \frac{1}{2} P_0(s)\right] P_k(s)\, ds$$

$$= t \left\{ \frac{1}{2\sqrt{3}} \underbrace{\int_0^1 P_1(s) P_k(s)\, ds}_{\delta_{k1}} + \frac{1}{2} \underbrace{\int_0^1 P_0(s) P_k(s)\, ds}_{\delta_{k0}} \right\}$$

$= 0$ for $k \geq 2$

<u>Case 2</u>: Consider the nonorthogonal, yet complete, basis $\{1, t, t^2, \ldots\}$.

We'll normalize the elements $g_k(t) = \frac{1}{\sqrt{2k+1}} t^k \qquad k = 0, 1, 2, \ldots$

$$L g_k(t) = \int_0^1 s t \frac{1}{\sqrt{2k+1}} s^k \, ds = \frac{1}{\sqrt{2k+1}} \frac{1}{k+2} t$$

Matrix elements of L in $\{g_k\}$ basis

$$l_{kl} = \langle L g_k, g_l \rangle = \frac{1}{\sqrt{2k+1}} \frac{1}{\sqrt{2l+1}} \frac{1}{(k+2)(l+2)} \qquad k, l \geq 0$$

$$l_{00} = \frac{1}{4} \qquad l_{01} = \frac{\sqrt{3}}{18} \qquad l_{10} = \frac{\sqrt{3}}{18} \qquad l_{22} = \frac{1}{27}$$

2×2 matrix rep$^{\underline{n}}$ $\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{18} \\ \frac{\sqrt{3}}{18} & \frac{1}{27} \end{bmatrix}$ Eigenvalues $\lambda_1 \cong 0.287\overline{037}$

$\lambda_2 = 0$

10×10 $\lambda_1 \cong 0.31324\ldots, 0, 0, 0, 0$ MAPLE

20×20 $\lambda_1 \simeq 0.31474\ldots, 0, 0, 0, \ldots$ "intsttn"

<u>Case 3</u> orthonormal basis $g_k = \sin k\pi t$

10×10 $\lambda_1 \cong 0.314048\ldots$ $\lambda_2 \ldots \lambda_{10}$: $|\lambda_i| < 10^{-10}$

20×20 $\lambda_1 \simeq 0.3235\ldots$ $|\lambda_i| < 10^{-10}$

30×30 $\lambda_1 \simeq 0.3267$ $|\lambda_i| < 10^{-10}$ MAPLE

$\downarrow$

$\frac{1}{3}$? "intstsin"

2. $$Lf = \int_0^1 (s-t)^2 f(s)\,ds$$

$L : f \to c_0 + c_1 t + c_2 t^2 \quad \Rightarrow \quad L$ is compact

In P_n basis
$$L = \begin{bmatrix} \begin{matrix} \frac{1}{6} & 0 & \frac{\sqrt{5}}{30} \\ 0 & -\frac{1}{6} & 0 \\ \frac{\sqrt{5}}{30} & 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$$

MAPLE "orthosmt2"

$$\lambda = 0,\ -\frac{1}{6},\ \frac{1}{12}+\frac{\sqrt{5}}{20},\ \frac{1}{12}-\frac{\sqrt{5}}{20}$$

$-0.1\dot{6} \quad \simeq 0.1951367 \quad \simeq -0.02847$

$\|L\| \le \frac{1}{\sqrt{15}} \quad (L^2)$

$\| \quad 0.2582$

<u>Another method</u>: We look for an eigenvector of form $f(t) = a + bt + ct^2$

$$Lf = \int_0^1 s^2 f(s)\,ds - 2t\int_0^1 s f(s)\,ds + t^2 \int_0^1 f(s)\,ds$$

$$= \left[\tfrac{1}{3} + \tfrac{1}{4}b + \tfrac{1}{5}c\right] - 2t\left[\tfrac{1}{2}a + \tfrac{1}{3}b + \tfrac{1}{4}c\right] + t^2\left[a + \tfrac{1}{2}b + \tfrac{1}{3}c\right]$$

$$Kf = \lambda f \Rightarrow \quad \lambda(a,b,c) = \left(\tfrac{1}{3} + \tfrac{1}{4}b + \tfrac{1}{5}c,\ -a - \tfrac{2}{3}b - \tfrac{1}{2}c,\ a + \tfrac{1}{2}b + \tfrac{1}{3}c\right)$$

$$\Rightarrow \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ -1 & -\frac{2}{3} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

eigenvalues $-\frac{1}{6},\ \frac{1}{12}+\frac{\sqrt{5}}{20},\ \frac{1}{12}-\frac{\sqrt{5}}{20}$ as found numerically above.