Addendum to p. 38

Example 3.7 (continued)

We'll modify the norm on X so that $D = \frac{d}{dt}$ is continuous

$$D : X = C^1[0,1] \to Y = C[0,1]$$

On X: $\|f\|_X = \max\{\|f\|_\infty, \|f'\|_\infty\}$ (see Ex. (5) p. 7 of Course Notes)

On Y: $\|g\|_Y = \|g\|_\infty$

This takes the derivative into consideration

Now consider $f_n = \frac{1}{\pi n^2} \sin n\pi t \Rightarrow \|f_n\|_X = \max\left\{\frac{1}{\pi n^2}, \frac{1}{n}\right\}$

$$= \frac{1}{n}$$

$$\|f_n\|_X = \frac{1}{n} \xrightarrow{n\to\infty} 0 \Rightarrow f_n \to 0$$

$$Df_n = \frac{1}{n}\cos n\pi t \Rightarrow \|Df_n\|_Y = \left\|\frac{1}{n}\cos n\pi t\right\|_\infty = \frac{1}{n}$$

$$\Rightarrow \|Df_n\|_Y \xrightarrow{n\to\infty} 0 \Rightarrow Df_n \to 0$$

$\Rightarrow D$ is cts at $0 \Rightarrow D$ is cts at all $f \in X$

Let $g_n(t) = \frac{1}{n\pi} \sin n\pi t \Rightarrow \|g_n\|_X = \max\left\{\frac{1}{n\pi}, 1\right\} = 1$

$$Dg_n = \cos n\pi t \Rightarrow \|Dg_n\|_Y = \|\cos n\pi\|_\infty = 1$$

$\Rightarrow \|D\| \leq 1$ $\quad$ D is now a bounded operator.

Milroy

This leads to the following natural infinity norm on the space $C^n[a,b]$ of k-times continuously differentiable functions on $[a,b]$, ie

$$C^n[a,b] = \{ f : [a,b] \to \mathbb{R},\ f^{(n)} \in C[a,b] \}$$

where $f^{(k)} = \frac{d^k}{dt^k} f(t)$ (Note that $f^{(k)} \in C[a,b] \Rightarrow f^{(k-1)} \in C[a,b]$)

$$\| f \|_{n,\infty} = \max_{0 \le k \le n} \{ \| f^{(k)} \|_\infty \}$$

eg. $f(t) = \sin 5t$ on $[-\pi, \pi]$

$$\| f^{(k)} \|_\infty = 5^k \qquad k = 0, 1, 2, \ldots$$

so

$$\| f \|_{n,\infty} = 5^n \qquad n = 0, 1, 2, \ldots$$

This norm defines the following metric on $C^n[a,b]$

$$d_{n,\infty}(f,g) = \| f - g \|_{n,\infty} = \max_{1 \le k \le n} \{ \| f^{(k)} - g^{(k)} \|_\infty \}$$

Hilroy