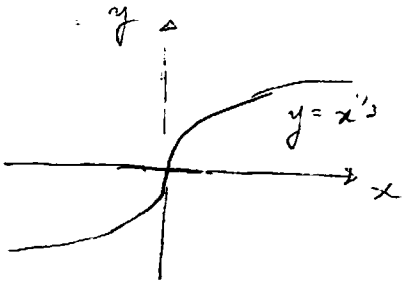
Addendum to p. 15

Consider $\frac{dx}{dt} = x^{1/3}$ $\quad x(0) = x_0$

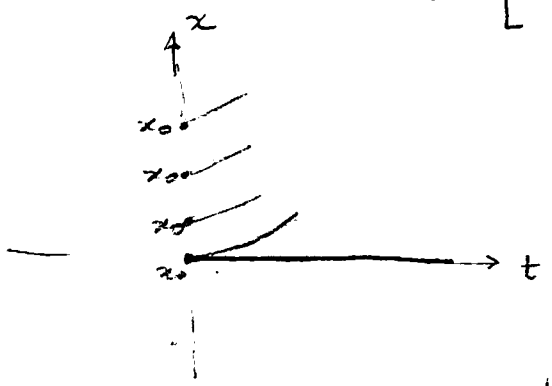


$\frac{dy}{dx} \to \infty$ as $x \to 0$

$$\frac{dx}{x^{1/3}} = dt \qquad \frac{3}{2}\left[x^{2/3} - x_0^{2/3}\right] = t$$

$$\int_0^t \frac{dx}{x^{1/3}} = \int_0^t ds \qquad x(t)^{2/3} - x_0^{2/3} = \frac{2}{3}t$$

$$x(t) = \left[x_0^{2/3} + \frac{2}{3}t\right]^{3/2}$$



For $x_0 \neq 0$, there is a unique solution $x(t)$

When $x_0 = 0$, the above method gives

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}$$

But there is <u>another</u> solution, $x(t) = 0$

Check $\frac{dx}{dt} = 0 \qquad x^{1/3}(t) = 0.$

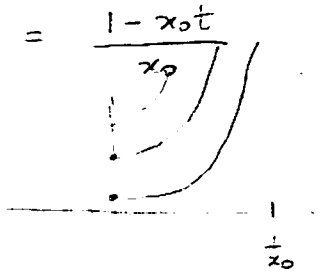
When $x_0 = 0$, solution is not unique.

This is because $x^{1/3}$ is "non-Lipschitz" at $x = 0$.

Contrast with $\frac{dx}{dt} = x^2 \qquad \frac{dx}{x^2} = dt \qquad -\frac{1}{x(t)} + \frac{1}{x_0} = t \qquad x_0 \neq 0$

$$\frac{1}{x(t)} = \frac{1}{x_0} - t = \frac{1 - x_0 t}{x_0}$$

$$x(t) = \frac{x_0}{1 - x_0 t}$$



When $x_0 = 0$. $x(t) = 0$ is unique solution