Lecture 5

Series solutions to DEs

`Relevant sections from AMATH 351 Course Notes (Wainwright): 1.4.1`

`Relevant sections from AMATH 351 Course Notes (Poulin and Ingalls): 2.1-2.3`

As mentioned earlier in this course, linear second order DEs of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{1}$$

where the $a_i(x)$ are polynomials in x, are encountered in many applications (e.g., Laplace's equation $\nabla^2 u = 0$, separation of variables, etc.). Such DEs are rarely solvable in "closed form," i.e., in terms of standard functions. It would seem reasonable to seek solutions to (1) that involve powers of x. One could try single powers of x but for polynomial $a_i(x)$, this probably wouldn't work. For the same reason, assuming $y(x)$ to be a polynomial in x would probably not work in general. So we might resort to "infinite polynomials," i.e., power series expansions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{2}$$

where x_0 is the point of expansion. (In many applications, $x_0 = 0$.) By doing this, we are assuming that the solution $y(x)$ to our DE has a Taylor series expansion. Sometimes this will work; other times it won't, as we'll see.

For the moment, we'll consider a couple of DEs with the particularly simple form,

$$y'' + P(x)y' + Q(x)y = 0\,, \tag{3}$$

where $P(x)$ and $Q(x)$ are polynomial functions of x. Since they are polynomial functions, they do not "blow up" at any $x \in \mathbb{R}$. And note that the first term y'' is not multiplied by anything. As such, the above DE has no singular points. We shall assume solutions to Eq. (3) having the form,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \,, \tag{4}$$

i.e., simple power series solutions.

Example 1: Consider the DE,

$$y'' + y = 0\,. \tag{5}$$

(Of course, we know that two linearly independent solutions to this DE are $y_1(x) = \cos x$ and $y_2(x) = \sin x$, but let's see what the power series in (4) can give us.)

We shall try two approaches to detemine power series solutions. The first approach involves a straightforward substitution of the power series in (4) into the DE (5) where we write out the expansions, at least to the first few terms. This approach will give an idea of how the method works, at least for the first few terms. The second approach, a more mathematical one, will be performed using sigma notation. In general, it provides a general relation between the coefficients a_n, from which the series can be constructed.

Method No. 1: Writing out the power series explicitly to the first few terms

We start with the power series expansion assumed for the solution $y(x)$, written out to the first few terms, i.e.,

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \,. \tag{6}$$

We'll assume that this series has a nonzero radius of convergence, i.e., it converges for $|x| < R$ where $R > 0$ is the radius of convergence. (We don't worry about the actual value of R for the moment and simply let the mathematics take us to the final result.) If the series converges for $|x| < R$, then the derivative $y'(x)$ will admit an expansion that is produced by differentiating the series for $y(x)$ termwise, i.e.,

$$y'(x) = a_1 x + 2a_2 x + 3a_3 x^2 + \cdots \,. \tag{7}$$

From 1B Calculus, this series will also converge for $|x| < R$. We differentiate one more time to produce the series expansion for $y''(x)$, since it is needed in Eq. (5),

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots . \tag{8}$$

We now substitute the series for $y(x)$ and $y''(x)$ into (5):

$$(2a_2 + 6a_3x + 12a_4x^2 + \cdots) + (a_0 + a_1x + a_2x^2 + \cdots) = 0 . \tag{9}$$

This is an equation that must be satisfied for all $|x| < R$, not just at one value of x. The problem is that it involves an infinite number of powers of x^n. But that is not really a problem. We first collect terms in like powers of x - essentially a rearrangement of terms in the above equation,

$$(2a_2 + a_0) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + \cdots = 0 . \tag{10}$$

Because of the linear independence of the functions x^n, $n = 0, 1, 2, \cdots$, the above equation is satisfied for all $|x| < R$ only if the coefficient of each power of x is zero. This leads to the following equations,

$$2a_2 + a_0 = 0 , \tag{11}$$

$$6a_3 + a_1 = 0 , \tag{12}$$

$$12a_4 + a_2 = 0 . \tag{13}$$

This might look like a formidable system of equations to solve, but it's not really that bad. Note that we can rewrite Eq. (11) as follows,

$$a_2 = -\frac{1}{2}\, a_0 . \tag{14}$$

And Eq. (12) can be rewritten as follows,

$$a_3 = -\frac{1}{6}a_1 . \tag{15}$$

Finally, Eq. (13) yields,

$$a_4 = -\frac{1}{12}a_2 . \tag{16}$$

Notice that each coefficient a_n is determined by the coefficient a_{n-2}. The reader might even be able to see the general pattern in the above equations, i.e.,

$$a_n = -\frac{1}{n(n-1)} a_{n-2}\,, \quad n = 2, 3, 4, \cdots . \tag{17}$$

We might also wish to "bump up" the index n by 2, i.e., $n \to n+2$ to yield

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n\,, \quad n = 0, 1, 2, \cdots . \tag{18}$$

This is an example of a **recursion relation** between the coefficients a_n. In this particular case, the recursion relation shows that the coefficient a_0 determines a_2 which, in turn, determines a_4. In fact,

$$a_4 = -\frac{1}{12} a_2 = -\frac{1}{12}\left[-\frac{1}{2} a_0\right] = \frac{1}{24} a_0\,. \tag{19}$$

If we set $n = 3$ in Eq. (18), then we obtain a_5:

$$a_5 = -\frac{1}{20} a_3 = -\frac{1}{20}\left[-\frac{1}{6} a_1\right] = \frac{1}{120} a_1\,. \tag{20}$$

We see that all even-indexed coefficients a_n, n even, can be expressed in terms of a_0. As well, all odd-indexed coefficients a_n, n odd, can be expressed in terms of a_1. So what are the values of a_0 and a_1? The answer is that there are no particular values for a_0 and a_1 – they are arbitrary! That should be reminiscent of the arbitrary constants that are involved in the general solution of a homogeneous DE. If we use the above results to express the coefficients a_2-a_5 in terms of a_0 and a_1 in the in the power series expansion in (6), we obtain

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^2 + \cdots\right] + a_1\left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right]. \tag{21}$$

If we rewrite this equation as

$$y(x) = a_0\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^2 + \cdots\right] + a_1\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right]. \tag{22}$$

the reader may recognize that the method has generated the first three terms in the Taylor series expansions of $\cos x$ and $\sin x$, i.e.,

$$y(x) = a_0 \cos x + a_1 \sin x\,. \tag{23}$$

Of course, this is what we were hoping for since we know that it is the general solution of the DE in (5). In other words, our series solution method has produced the general solution.

Method No. 2: Using sigma notation

We'll now employ sigma notation and show that it will produce the general recursion relation between series coefficients a_n – no guessing is necessary. We start with the series expansion for $y(x)$,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \,. \tag{24}$$

The series for $y'(x)$ is obtained by termwise differentiation, which means that we simply compute the derivative of $a_n x^n$:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \,. \tag{25}$$

Note that the summation now starts at $n = 1$, which is in agreement with Eq. (7), where we wrote out the first few terms of $y'(x)$ explicitly. Differentiating one more time yields the series for $y''(x)$,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \,, \tag{26}$$

which is in agreement with Eq. (7).

The next step is to substitute the expansions in (24) and (26) into the DE in Eq. (5). As you'll recall from Method No. 1, we shall eventually have to collect terms having like powers of x^n. For this reason, it is necessary to "bump up" the index and power $n-2$ in the series expansion for $y''(x)$ to the index and power n. We could simply replace n with $n+2$ in all places in Eq. (26) but then remains the question of determining the lowest index. Just to make sure that we do things correctly, we can define a new index,

$$k = n - 2 \qquad \text{which implies that} \qquad n = k + 2 \,. \tag{27}$$

We then replace n with $k+2$ in (26):

$$y''(x) \sum_{?}^{\infty} (k+2)(k+1) a_{k+2} x^k \,. \tag{28}$$

The "?" means that we have to determine the lowest possible value of k. Since the lowest index of n in Eq. (26) is 2, it follows, from $k = n-2$, that the lowest index of k in (28) is $k = 0$, i.e.,

$$y''(x) \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \,. \tag{29}$$

Finally, we replace k with n so that all powers of x are in x^n:

$$y''(x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\,. \tag{30}$$

We now substitute Eqs. (24) and (30) into the DE in (5) to yield

$$y''+y=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_nx^n=0\,. \tag{31}$$

We can then combine the two summations into one summation, which is essentially collecting terms in like powers of x^n:

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0\,. \tag{32}$$

Since this equation must hold for all $|x|<R$, it is necessary that the coefficient of each term in x^n is zero, i.e.,

$$(n+2)(n+1)a_{n+2}+a_n=0\,,\quad n=0,1,2,\cdots\,. \tag{33}$$

At this point, note that Eq. (33) is in agreement with Eqs. (11), (12) and (13), which were obtained by writing out the first few terms of the series explicitly. An advantage of the sigma notation method is that it yields the general relation without having to guess it from particular cases. From Eq. (33), we have the following recursion relation for the a_n,

$$a_{n+2}=-\frac{1}{(n+2)(n+1)}a_n\quad n=0,1,2,\cdots\,, \tag{34}$$

which is in agreement with Eq. (18) obtained in Method No. 1.

Let us once again compute a_2, a_4 and a_6 from a_0 using the above recursion relation. Choosing $n=0$, $n=2$ and $n=4$ yields,

$$\begin{aligned}
a_2 &= -\frac{1}{2\cdot 1}a_0=-\frac{1}{2!}a_0\\
a_4 &= -\frac{1}{4\cdot 3}a_2=\frac{1}{4\cdot 3\cdot 2\cdot 1}a_0=\frac{1}{4!}a_0\\
a_6 &= -\frac{1}{6\cdot 5}a_4=\frac{1}{4\cdot 3\cdot 2\cdot 1}a_2=-\frac{1}{6!}a_0\,.
\end{aligned} \tag{35}$$

The reader should be able to determine the general pattern for even-indexed coefficients,

$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \, , \quad k = 1, 2, 3, \cdots . \tag{36}$$

Once again a_0 is arbitrary.

We may also compute a_3, a_5 and a_7 from a_1 by choosing $n = 1$, $n = 3$ and $n = 5$ in (34),

$$\begin{aligned} a_3 &= -\frac{1}{3 \cdot 2} a_1 = -\frac{1}{3!} a_1 \\ a_5 &= -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_1 = \frac{1}{5!} a_1 \\ a_7 &= -\frac{1}{7 \cdot 6} a_5 = \frac{1}{7 \cdot 6 \cdot 5 \cdot 4} a_3 = -\frac{1}{7!} a_0 \, . \end{aligned} \tag{37}$$

In general, for odd-indexed coefficients,

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1 \, , \quad k = 1, 2, 3, \cdots . \tag{38}$$

The series expansion in (6) may then be grouped into even and odd powers in the form

$$y(x) = \sum_{k=1}^{\infty} a_{2k} x^{2k} + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1} \, , \tag{39}$$

which, using Eqs. (36) and (38), yields

$$\begin{aligned} y(x) &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ &= a_0 y_1(x) + a_1 y_2(x) \, , \end{aligned} \tag{40}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions. Neither function is a constant multiple of the other. In fact, $y_1(x)$ is an even function of x and $y_2(x)$ is an odd function. Recalling either the results of Method No. 1 or 1B Calculus, we recognize these series expansions as Taylor series expansions of $\sin x$ and $\cos x$, respectively, i.e.,

$$y_1(x) = \sin x \, , \qquad y_2(x) = \cos x \, . \tag{41}$$

Finally, using the Ratio Test it can be verified (Exercise) that both series converge for all $x \in \mathbb{R}$.

Some theoretical matters

Here we make some comments about the theoretical basis of the series expansion method. The first step is to once again recast Eq. (1) into standard form,

$$y'' + P(x)y' + Q(x)y = 0, \tag{42}$$

where

$$P(x) = \frac{a_1(x)}{a_2(x)}, \qquad Q(x) = \frac{a_0(x)}{a_2(x)}. \tag{43}$$

We see that if the $a_k(x)$ are polynomials in x, then $P(x)$ and $Q(x)$ could be rational functions in x. As such, we may have to worry about points for which $a_2(x) = 0$.

The point $x = x_0$ is an **ordinary point** of Eq. (42) if $P(x)$ and $Q(x)$ are *analytic* at x_0, that is, they possess Taylor series expansions of the form

$$P(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \qquad Q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n, \tag{44}$$

that both have nonzero radii of convergence. (This means that all derivatives of P and Q exist at x_0.) If x_0 is not an ordinary point, then it is a **singular point**. We shall be looking at singular points in a little while.

There is an important theorem – see the book by Simmons, Theorem A on page 155 – that treats the case of series expansions about ordinary points:

Theorem: If there exists an $R > 0$ such that both series for $P(x)$ and $Q(x)$ converge for all $|x - x_0| < R$, then the ODE in (42) possesses two linearly independent power series solutions that converge for all $|x - x_0| < R$. It follows that if $P(x)$ and $Q(x)$ are both polynomials, hence analytic for all $x \in \mathbb{R}$, then the series solutions must converge for all $x \in \mathbb{R}$.

Example 2: We now consider the DE,

$$y'' + 2x^2y' + xy = 0. \tag{45}$$

Here $P(x) = 2x^2$ and $Q(x) = x$ are both analytic everywhere. We shall choose $x_0 = 0$ and assume series solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{46}$$

We expect the series solutions to converge for all $x \in \mathbb{R}$.

In what follows, a rather simplified method of substitution of power series will be used. It will save some work in that we won't worry about where each series starts. We allow for the appearance of coefficients with negative indices a_k for $k < 0$ in our recurrence relations for the a_n but simply assume that such coefficients are zero, i.e., we assume that

$$a_k = 0 \quad \text{for} \quad k < 0\,. \tag{47}$$

To illustrate, we take the formal derivatives of the above power series and disregard the actual numerical value of the lower limit of summation:

$$y(x) = \sum_n a_n x^n, \quad y'(x) = \sum_n n a_n x^{n-1}, \quad y''(x) = \sum_n n(n-1) a_n x^{n-2}. \tag{48}$$

Substitution into the DE yields

$$\sum_n n(n-1) a_n x^{n-2} + \sum_n 2n a_n x^{n+1} + \sum_n a_n x^{n+1} = 0 \tag{49}$$

We then collect like powers of x^n, "bumping up" or "bumping down" indices within each summation in order to produce the term x^n. For example, in order to produce x^n in the first summation, we have to replace each n with $n+2$. (This is equivalent to setting $k = n-2$, implying $n = k+2$, etc.. Too much work!) The net result is the following summation:

$$\sum_n [(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + a_{n-1}]x^n = 0, \tag{50}$$

which can be further simplified to

$$\sum_n [(n+2)(n+1)a_{n+2} + (2n-1)a_{n-1}]x^n = 0, \tag{51}$$

We therefore have the difference equation

$$(n+2)(n+1)a_{n+2} + (2n-1)a_{n-1} = 0, \qquad \text{for all } n \in \mathbf{Z}. \tag{52}$$

For example, we could set $n = -20,000$, but both terms would be zero. It's only when we get close to zero that interesting things happen. If we let $n = -1$, then the above equation becomes

$$0 \cdot a_{-1} + (-3)a_{-2} = 0 \quad \Longrightarrow \quad 0 = 0\,. \tag{53}$$

Clearly, this is not telling us anything. But when we set $n = 0$, Eq. (52) becomes

$$2a_2 + (-1)a_{-1} = 0. \tag{54}$$

But $a_{-1} = 0$, implying that $a_2 = 0$. When $n = 1$,

$$6a_3 + a_0 = 0, \text{ implying that } a_3 = -\frac{a_0}{6}. \tag{55}$$

For $n \geq 1$, (52) can be rearranged to give

$$a_{n+2} = -\frac{2n-1}{(n+2)(n+1)}a_{n-1}, \quad n = 1, 2, \cdots. \tag{56}$$

And if you want to make this recurrence relation look a little "nicer," you can bump up the indices by 1 to give

$$a_{n+3} = -\frac{2n+1}{(n+3)(n+2)}a_n, \quad n = 0, 1, \cdots. \tag{57}$$

We now see that a_0 determines a_3 which determines a_6, etc.. And a_1 determines a_4 which determines a_7, etc.. Since $a_2 = 0$, it follows that $a_5 = a_8 = \cdots = 0$. With a little more work, we can generate the next few elements of each series. As a result, we obtain the following two power series expansions:

$$a_0\left[1 - \frac{1}{6}x^3 + \frac{7}{180}x^6 - \cdots\right] \tag{58}$$

and

$$a_1\left[x - \frac{1}{4}x^4 + \frac{3}{56}x^7 - \cdots\right]. \tag{59}$$

These two series are linearly independent so that we can write the general solution of the DE as

$$y(x) = a_0\left[1 - \frac{1}{6}x^3 + \frac{7}{180}x^6 - \cdots\right] + a_1\left[x - \frac{1}{4}x^4 + \frac{3}{56}x^7 - \cdots\right]. \tag{60}$$

Suppose we are given the initial condition $y(0) = 1$ and $y'(0) = 1$. The first condition implies that $a_0 = 1$. The second implies that $a_1 = 1$. The solution to this initial value problem is the function defined by

$$y(x) = 1 + x - \frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{7}{180}x^6 - \cdots . \tag{61}$$

For reasons stated earlier, this series will converge for all $x \in \mathbb{R}$.

Exercise: Study Example 1.6 in the AMATH 351 Course Notes by J. Wainwright, p. 32.

Special case: Legendre's differential equation

(This section was not covered in class and is presented for information. It may be helpful in Problem Set No. 2.)

Let us now consider the so-called *Legendre* equation,

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \quad p \in \mathbf{R}, \tag{62}$$

which occurs quite frequently in applied mathematics and theoretical physics. (In the solution of the Laplace and Schrödinger equations in three dimensions, x above represents the angular coordinate θ. In quantum mechanics, the constant p will represent the orbital angular momentum quantum number.) Rewriting this DE in standard form,

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0, \tag{63}$$

we have

$$P(x) = -\frac{2x}{1-x^2}, \quad Q(x) = \frac{p(p+1)}{1-x^2}. \tag{64}$$

The functions $P(x)$ and $Q(x)$ are seen to be analytic at $x = 0$. The "bad" points of this DE are $x = \pm 1$. As a consequence, the power series of $P(x)$ and $Q(x)$ have power series expansions about 0 with radius of convergence $R = 1$. For example,

$$P(x) = \frac{2x}{1-x^2} = 2x[1 + x^2 + x^4 + \cdots], \quad |x| < 1. \tag{65}$$

From the theorem cited in the previous lecture, we expect that power series solutions of (1) having the form

$$y(x) = \sum_n a_n x^n \tag{66}$$

will also have radii of convergence $R = 1$.

Let us now work out these series solutions. We substitute the above expansion into the DE in (1) (not paying attention to the lower limits of summation, with the understanding that $a_n = 0$ for $n < 0$):

$$\sum n(n-1)a_n x^{n-2} - x^2 \sum n(n-1)a_n x^{n-2} - 2x \sum n a_n x^{n-1} + p(p+1) \sum a_n x^n = 0. \tag{67}$$

Collecting like terms in x^n, and bumping up or down the indices of each summation as necessary:

$$\sum[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n]x^n = 0, \tag{68}$$

which can be simplified further to

$$\sum[(n+2)(n+1)a_{n+2} + [-n(n+1) + p(p+1)]a_n]x^n = 0. \tag{69}$$

This implies that

$$(n+2)(n+1)a_{n+2} + [-n(n+1) + p(p+1)]a_n = 0 \tag{70}$$

for all n. Setting $n = 0$ gives

$$2a_2 + p(p+1)a_0 = 0, \quad \text{or} \quad a_2 = -\frac{1}{2}p(p+1)a_0. \tag{71}$$

Setting $n = 1$ gives

$$6a_3 + [-2 + p(p+1)a_1 = 0, \quad \text{or} \quad a_3 = -\frac{1}{6}(p-1)(p+2)a_1. \tag{72}$$

For general $n \geq 0$,

$$\begin{aligned} a_{n+2} &= \frac{n(n+1) - p(p+1)}{(n+1)(n+2)}a_n \\ &= -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}a_n \end{aligned} \tag{73}$$

Let's compute the next two terms: For $n = 2$ in (73), we have

$$a_4 = -\frac{(p-2)(p+3)}{4 \cdot 3}a_2 = \frac{p(p-2)(p+1)(p+3)}{4!}a_0, \tag{74}$$

and for $n = 3$ we have

$$a_5 = -\frac{(p-3)(p+4)}{5 \cdot 4}a_2 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1, \tag{75}$$

We can express this result in the form

$$y(x) = a_0 L_p^0(x) + a_1 L_p^1(x), \tag{76}$$

where $L_p^0(x)$ is an even-powered series (hence defining an even function over its interval of convergence) and $L_p^1(x)$ is an odd-powered series (hence defining an odd function over its interval of convergence).

If p is a nonnegative integer, then one of the above series terminates and the other remains as an infinite series:

1. If p is even, then the series in $L_p^0(x)$ terminates, resulting in an even polynomial of degree p.

2. If p is odd, then the series in $L_p^1(x)$ terminates, resulting in an odd polynomial of degree p.

We tabulate the first few polynomials that result from p being a nonnegative integer:

1. $p = 0$: $L_0^0(x) = 1$,

2. $p = 1$: $L_1^1(x) = x$,

3. $p = 2$: $L_2^0(x) = 1 - 3x^2$,

4. $p = 3$: $L_3^1(x) = x - \frac{1}{3}x^3$.

These polynomials (up to a constant) are known as the *Legendre polynomials* and are extremely important in applied mathematics and theoretical physics. Such polynomial solutions can also exist for a number of other second order DEs encountered in physics, e.g., Laguerre, Hermite DEs.

Lecture 6

Series solutions at singular points – the method of Frobenius

`Relevant sections from AMATH 351 Course Notes (Wainwright): 1.4.2`

`Relevant sections from AMATH 351 Course Notes (Poulin and Ingalls): 2.4-2.5`

In many applications, one is forced to construct solutions to DEs about singular points. For example, $x = 0$ is a singular point of Bessel's equation – we'll discuss it in detail later – yet it is still a very convenient point around which to expand. There is still hope, for there are "bad" singular points and "not so bad" singular points. We shall be able to work with the latter.

Let us once again consider a second order linear ODE written in standard or normalized form,

$$y'' + P(x)y' + Q(x)y = 0. \tag{77}$$

Recall that x_0 is an **ordinary point** of (77) if $P(x)$ and $Q(x)$ are analytic at x_0: A function $f(x)$ is analytic at a point x_0 if it possesses a Taylor power series representations about x_0 of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \,. \tag{78}$$

This implies that $f(x)$ and all of its derivatives are defined at x_0.

We now consider the case that x_0 is not an ordinary point of (77), i.e., at least one of $P(x)$ and $Q(x)$ are not analytic at x_0. Then x_0 is a **singular point**. Suppose, however, that the functions

$$(x - x_0)P(x) \qquad \text{and} \qquad (x - x_0)^2 Q(x)$$

are analytic at x_0. Then x_0 is said to be a *regular singular point.* Otherwise, it is an *irregular singular point.* The main idea here is that for x_0 to be a regular singular point,

1. $P(x)$ behaves no worse than $\dfrac{1}{x - x_0}$ at x_0 and

2. $Q(x)$ behaves no worse than $\dfrac{1}{(x-x_0)^2}$ at x_0.

Examples:

1. The ODE
$$x^3y'' + x^2y' + y = 0.$$
In standard form, it becomes
$$y'' + \frac{1}{x}y' + \frac{1}{x^3} = 0.$$
Clearly $x_0 = 0$ is a singular point, with
$$P(x) = \frac{1}{x}, \qquad Q(x) = \frac{1}{x^3}.$$
But:

$xP(x) = 1$ is analytic at 0

$x^2Q(x) = \dfrac{1}{x}$ is not analytic at 0.

Therefore $x = 0$ is an **irregular singular point**.

2. The ODE
$$x^2y'' + xy' + y = 0.$$
In standard form, it becomes
$$y'' + \frac{1}{x}y' + \frac{1}{x^2} = 0.$$
Clearly $x_0 = 0$ is a singular point, with
$$P(x) = \frac{1}{x}, \qquad Q(x) = \frac{1}{x^2}.$$
But:

$xP(x) = 1$ is analytic at 0

$x^2Q(x) = 1$ is also analytic at 0.

Therefore $x = 0$ is a **regular singular point.**

In what follows, we shall consider only the case $x_0 = 0$ because it is the most commonly encountered singular point. If $x_0 = 0$ is a singular point, then a Taylor series expansion of the form,

$$y(x) = \sum_n a_n x^n \,, \tag{79}$$

will not generally work. However, if $x_0 = 0$ is a *regular* singular point, then a modified power series solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0, \tag{80}$$

where r is to be determined, may work. This is known as the *method of Frobenius.*

Note the condition that a_0 be nonzero in (80): This is to guarantee that the series starts somewhere. As before, we shall assume that all coefficients with negative indices are zero, i.e.,

$$a_n = 0 \quad \text{for} \quad n < 0\,. \tag{81}$$

Also note what the Frobenius method is doing: it is trying to accomodate for a possible singularity of the solution $y(x)$ to the DE at the singular point $x_0 = 0$. Near $x = 0$, $y(x)$ behaves approximately as

$$y(x) \approx x^r \quad \text{as} \;\; x \to 0. \tag{82}$$

If $r < 0$, then $y(x)$ is not continuous at 0. You have already encountered such a situation with Euler's equation,

$$x^2 y'' + pxy' + qy = 0\,, \tag{83}$$

which may admit solutions of the form $y(x) = x^r$.

Example: Consider the DE

$$4xy'' + y' + y = 0. \tag{84}$$

Here,

$$P(x) = \frac{1}{4x}, \qquad Q(x) = \frac{1}{4x},$$

so that $x = 0$ is a singular point. But

$$xP(x) = \frac{1}{4}, \qquad x^2Q(x) = \frac{x}{4},$$

which are both analytic at $x = 0$. Therefore $x = 0$ is a **regular singular point.**

We now substitute the Frobenius series (80) into the DE, differentiating termwise. (As before, we shall not bother to determine the lower indices of each summation. The condition that coefficients with negative indices are zero will take care of everything.)

$$y' = \sum (n+r)a_n x^{n+r-1}, \qquad y'' = \sum (n+r)(n+r-1)a_n x^{n+r-2}.$$

Substitution into the DE yields

$$4\sum (n+r)(n+r-1)a_n x^{n+r-1} + \sum (n+r)a_n x^{n+r-1} + \sum a_n x^{n+r} = 0.$$

We now collect terms in x^{n+r}, "bumping up" or "bumping down" indices in the various summations in order to obtain the proper power of x. For example, in order to produce x^{n+r} in the first summation, we must replace n with $n+1$. The same is true for the second summation. The third summation can be left alone. The result is

$$\sum [4(n+r+1)(n+r)a_{n+1} + (n+r+1)a_{n+1} + a_n]x^{n+r} = 0.$$

Since all coefficients of x^{n+r} must vanish (because this relation is to hold true for all x on some interval I), we have, after a little simplification,

$$(n+r+1)(4n+4r+1)a_{n+1} + a_n = 0, \qquad \text{for all } n. \tag{85}$$

Now if we use $n = -10,000$, this relation is trivially satisfied, since a_{-9999} and $a_{10,000}$ are both zero. But once we get to indices with values near zero, something happens. For $n = -2$, the LHS is automatically zero, since $a_{-1} = a_{-2} = 0$. But for $n = -1$, this relation becomes

$$r(4r-3)a_0 + a_{-1} = 0\,. \tag{86}$$

Recall that $a_{-1} = 0$. Because of the condition $a_0 \neq 0$, we have the following condition,

$$r(4r-3) = 0. \tag{87}$$

This is known as the *indicial equation.* The roots of this equation correspond to the values of r that we were looking for in the Frobenius solution (80). In this case, the roots are

$$r_1 = 0 \quad \text{and} \quad r_2 = \frac{3}{4}. \tag{88}$$

Each of the roots will generate a Frobenius solution. Once we've chosen a root r_i, we use the recursion relation (85) to generate the series coefficients.

Let us first rewrite the recursion relation (85) as follows,

$$a_{n+1} = -\frac{1}{(n+r+1)(4n+4r+1)} a_n, \quad n \geq 0. \tag{89}$$

We now consider the series which are defined for each of the two values of r, hoping that two linearly independent solutions will be generated.

Case 1: $r = 0$

The recursion relation in (89) becomes

$$a_{n+1} = -\frac{1}{(n+1)(4n+1)} a_n, \quad n \geq 0.$$

- $n = 0$: $a_1 = -a_0$
- $n = 1$: $a_2 = -\frac{1}{2 \cdot 5} a_1 = \frac{1}{2 \cdot 5} a_0$
- $n = 2$: $a_3 = -\frac{1}{3 \cdot 9} a_2 = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 9} a_0$

Thus the Frobenius series solution is, to four nonzero terms,

$$y_1(x) = a_0 \left[1 - x + \frac{1}{10} x^2 - \frac{1}{270} x^3 + \cdots \right]$$

With a little more work, we can determine the general form of the coefficients,

$$a_k = \frac{(-1)^k}{k!} \left[\Pi_{i=1}^{k} \frac{1}{4i-3} \right] a_0.$$

Going a little further, we can show that

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = 0,$$

implying that the series converges for all $x \in \mathbf{R}$.

Case 2: $r = \dfrac{3}{4}$. The recursion relation in (89) becomes, after simplification,

$$a_{n+1} = -\frac{1}{(n+1)(4n+7)} a_n, \quad n \geq 0.$$

- $n = 0$: $a_1 = -\dfrac{1}{7} a_0$
- $n = 1$: $a_2 = -\dfrac{1}{2 \cdot 11} a_1 = \dfrac{1}{2 \cdot 7 \cdot 11} a_0$
- $n = 2$: $a_3 = -\dfrac{1}{3 \cdot 15} a_2 = -\dfrac{1}{2 \cdot 3 \cdot 7 \cdot 11 \cdot 15} a_0$

Thus the Frobenius series solution is, to four nonzero terms,

$$y(x) = a_0 x^{3/4} \left[1 - \frac{1}{7} x + \frac{1}{154} x^2 - \frac{1}{6930} x^3 + \cdots \right]$$

The general form of the coefficients is

$$a_k = \frac{(-1)^k}{k!} \left[\Pi_{i=1}^{k} \frac{1}{4i+3} \right] a_0,$$

from which it follows that

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = 0,$$

This series also converges for all $x \in \mathbf{R}$.

The two series yielded by the roots $r_1 = 0$ and $r_2 = \dfrac{3}{4}$ are linearly independent. Therefore, the general solution of this DE is given by

$$y(x) = c_1 \left[1 - x + \frac{1}{10} x^2 - \frac{1}{270} x^3 + \cdots \right] + c_2 x^{3/4} \left[1 - \frac{1}{7} x + \frac{1}{154} x^2 - \frac{1}{6930} x^3 + \cdots \right] \quad (90)$$

Remark: Notice how we arrived naturally at the indicial equation (87) by collecting terms in x^{n+r} to produce relation (89) and then setting n appropriately to isolate the nonzero a_0

coefficient. Working in this way, one avoids having to memorize the indicial equation (1.119) presented in the Course Notes by J. Wainwright on p. 33.

It is not always the case that the Frobenius method will yield two linearly independent solutions. For example, consider the DE

$$4x^2y'' - 8xy' + (x+5)y = 0. \tag{91}$$

The standard form of this DE is

$$y'' - \frac{2}{x}y' + \frac{x+5}{4x^2}y = 0\,. \tag{92}$$

Here,

$$P(x) = -\frac{2}{x}\,, \quad Q(x) = \frac{(x+5)}{4x^2}\,. \tag{93}$$

The point $x_0 = 0$ is a singular point since $P(0)$ and $Q(0)$ are undefined. However,

$$xP(x) = -2 \quad \text{and} \quad x^2Q(x) = \frac{1}{4}(x+5) \tag{94}$$

are analytic at 0, so $x_0 = 0$ is a **regular singular point**.

If we assume a Frobenius series solution of the form

$$y(x) = x^r \sum_n a_n x^n = \sum_n a_n x^{n+r}, \qquad a_0 \neq 0, \tag{95}$$

we arrive at the recursion relation (Exercise):

$$[4(n+r)(n+r-1) - 8(n+r) + 5]a_n + a_{n-1} = 0. \tag{96}$$

Setting $n = -1$ gives the indicial equation

$$4r^2 - 12r + 5 = 0, \tag{97}$$

which has roots $r_1 = \dfrac{1}{2}$ and $r_2 = \dfrac{5}{2}$.

Case 1: $r = \frac{5}{2}$. The recursion relation in (96) becomes

$$4n(n+2)a_n + a_{n-1} = 0, \tag{98}$$

which can be rearranged to give

$$a_n = -\frac{1}{4n(n+2)}a_{n-1}, \quad n \geq 1. \tag{99}$$

This recursion yields a series of the form

$$y_1(x) = a_0 x^{5/2}\left[1 - \frac{1}{12}x + \frac{1}{384}x^2 - \cdots\right] \tag{100}$$

With a little more work, we can determine the general form of the coefficients,

$$a_k = \frac{(-1)^k}{k!}\frac{1}{2^{2k-1}k!(k+2)!}a_0. \tag{101}$$

and establish that the series converges for all $x \in \mathbf{R}$.

Case 2: $r = \frac{1}{2}$. The recursion relation in (96) becomes

$$4n(n-2)a_n + a_{n-1} = 0, \tag{102}$$

which can be rearranged to give

$$a_n = -\frac{1}{4n(n-2)}a_{n-1}, \quad n \geq 1. \tag{103}$$

This looks problematic: When $n = 2$, the denominator on the RHS is zero. Thus it appears that the method breaks down. Another way to see this is to work with relation (102):

- $n = 0$: $0 \cdot a_0 + a_{-1} = 0$. This is automatically satisfied by any a_0.
- $n = 1$: $-4 \cdot a_1 + a_0 = 0$, implying that $a_1 = -a_0/4$. So far, so good.
- $n = 2$: $0 \cdot a_2 + a_1 = 0$. This implies that $a_1 = 0$ which, from the $n = 1$ case, implies that $a_0 = 0$, which contradicts the original Frobenius assumption that $a_0 \neq 0$.

Therefore, the root $r = \dfrac{1}{2}$ does not yield a solution. But all is not lost, for we can always generate a second linearly indpendent solution using the reduction of order method. Recall that we assume a second solution of the form $y_2(x) = u(x)y_1(x)$, where y_1 is a known solution, in this case our Frobenius solution for $r = \dfrac{5}{2}$. The method yields the following equation for $u'(x)$:

$$u'(x) = \frac{1}{y_1(x)^2} e^{-\int P(x)dx} \tag{104}$$

In this case $P(x) = -\dfrac{2}{x}$ so that the above equation becomes

$$u'(x) = \frac{x^2}{y_1(x)^2}. \tag{105}$$

We can now formally substitute the series for $r = \dfrac{5}{2}$ into the above expression to produce a series expansion for $u'(x)$:

$$\begin{aligned} u'(x) &= \frac{x^2}{[x^{5/2}\sum_{n=0}^{\infty} a_n x^n]^2} \\ &= \frac{1}{x^3}\frac{1}{[\sum_{n=0}^{\infty} a_n x^n]^2} \\ &= x^{-3}[c_0 + c_1 x + c_2 x^2 + \cdots], \end{aligned}$$

where we have formally inverted the square of the $a_n x^n$ series. Here $c_0 = 1/a_0^2$. Antidifferentiation produces the following result for $u(x)$ (setting the arbitrary constant $C = 0$):

$$u(x) = -\frac{c_0}{2x^2} - \frac{c_1}{x} + c_2 \ln x + c_3 x + \cdots .$$

Remember that we then have to multiply this series by our y_1 series to give the second, linearly independent solution:

$$\begin{aligned} y_2(x) &= u(x)y_1(x) \\ &= \left[-\frac{c_0}{2x^2} - \frac{c_1}{x} + c_2 \ln x + c_3 x + \cdots\right] x^{5/2} \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Note: The appearance of the logarithmic term in $y_2(x)$ explains why we could not generate the general solution of this DE with the Frobenius method alone. Now, you may ask, what if one tried a solution of the form

$$y(x) = x^r \ln x \sum_{n=0}^{\infty} a_n x^n \quad ?$$

This is the basis of *asymptotic analysis*, which seeks to determine the so-called *dominant* behaviour of a function near a point, in this case, a singular point of a DE. Our graduate course AMATH 743 deals with this subject.

To summarize, we simply state what can or cannot be accomplished by the Frobenius method at a regular singular point. Given that the indicial equation has two roots, r_1 and r_2:

- If $r_1 - r_2$ is not an integer (which also implies that $r_1 \neq r_2$), then the Frobenius method yields two linearly indpendent solutions.
- If $r_1 - r_2$ is an integer, there are two cases:
  - If $r_1 = r_2$, then only one solution is obtained.
  - If $r_2 \neq r_2$, then at least one solution is guaranteed.

If only one solution is obtained, we may use reduction of order to find another linearly independent solution.

Lecture 7

An important application – Bessel's equation

`Relevant sections from AMATH 351 Course Notes (Wainwright): 1.5`

`Relevant sections from AMATH 351 Course Notes (Poulin and Ingalls): 2.6-2.10`

Recall Bessel's equation of order p:

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \geq 0. \tag{106}$$

In normal form, it becomes

$$y'' + \frac{1}{x}y' + \left(1 - \frac{p^2}{x^2}\right) y = 0, \tag{107}$$

so that

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{p^2}{x^2}. \tag{108}$$

from which we see that $x = 0$ is a singular point. Since

$$xP(x) = 1 \quad \text{and} \quad x^2Q(x) = x^2 - p^2, \tag{109}$$

are both analytic at $x_0 = 0$, it is a **regular singular point**. We may therefore assume a Frobenius-type series solution of the form

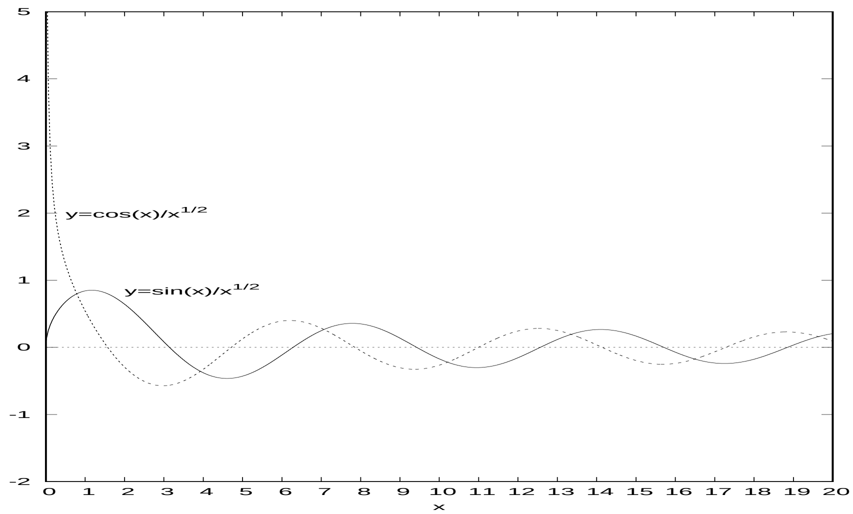
$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0. \tag{110}$$

Before our discussion on series solutions to Bessel's equation, recall that you have already seen one particular set of solutions to Bessel's DE. In Problem Set No. 1, you showed that

$$y_1(x) = \frac{1}{\sqrt{x}} \sin x, \quad y_2(x) = \frac{1}{\sqrt{x}} \cos x, \tag{111}$$

were linearly independent the particular case $p = \frac{1}{2}$. The graphs of these two functions are shown below over the interval $[0, 20]$.

As we'll see later, solutions to Bessel's DE for all $p \in \mathbb{R}$ exhibit some kind of oscillatory behaviour. (We shall also show later that, in general, the zeros two linearly independent



solutions $y_1(x)$ and $y_2(x)$ interlace each other, i.e., each zero of one solution is found between two consecutive zeros of the other, and vice versa.)

One final set of comments before we discuss series solutions to Bessel's DE: If we assume a solution to Bessel's DE in (106) of the form

$$y(x) = \frac{1}{\sqrt{x}} u(x)\,, \tag{112}$$

then, after some algebraic manipulation, we find that $u(x)$ satisfies the following DE,

$$u'' + \left(1 + \frac{1 - 4p^2}{4x^2}\right) u = 0\,. \tag{113}$$

(Why we employ $\frac{1}{\sqrt{x}}$ in (112) will be discussed in the next section.)

Note that as $x \to \infty$, the $\frac{1}{x^2}$ in (113) becomes negligible so that solutions to (113) will be well approximated by solutions to the DE,

$$v'' + v = 0\,. \tag{114}$$

Of course, the general solution to (114) is

$$v(x) = c_1 \sin x + c_2 \cos x\,. \tag{115}$$

Combining this result with Eq. (112), we claim that for very large x, any solution to Bessel's DE in (106) behaves as

$$y(x) \cong c_1 \frac{1}{\sqrt{x}} \sin x + c_2 \frac{1}{\sqrt{x}} \cos x\,, \tag{116}$$

for suitable values of c_1 and c_2. Note that in the special case $p = \frac{1}{2}$, the DE in (113) is **exactly** the DE in (114) so that the approximation in (116) becomes an equality, which is in agreement with Eq. (111).

Let us now return to an investigation of Frobenius series solutions of Bessel's DE having the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0. \tag{117}$$

Termwise differentiation and substitution of the resulting series into the DE in (106), followed by a collection of like terms in x^{n+r} yields

$$\sum [(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} - p^2 a_n] x^{n+r} = 0\,. \tag{118}$$

After a some additional algebra, we have the result,

$$[(n+r)^2 - p^2]a_n + a_{n-2} = 0\,. \tag{119}$$

Setting $n = 0$ gives

$$[r^2 - p^2]a_0 = 0, \tag{120}$$

since $a_2 = 0$. And since $a_0 \neq 0$, we have

$$r^2 - p^2 = 0, \tag{121}$$

the indicial equation associated with Bessel's DE of order p. The roots of this equation are

$$r_1 = p \quad \text{and} \quad r_2 = -p\,. \tag{122}$$

This implies that

$$r_1 - r_2 = 2p\,. \tag{123}$$

From the summary of the previous lecture we can conclude the following:

- If $r_1 - r_2 = 2p$ is not an integer, then we can obtain two linearly independent Frobenius solutions,

$$y_1(x) = x^p \sum a_n x^n, \qquad y_2(x) = x^{-p} \sum a_n x^n\,.$$

- If $r_1 - r_2 = 2p$ is an integer, then we can obtain at least one solution. If we can't obtain a second one using the Frobenius method, we can generate a second solution using the method of reduction of order applied to our one Frobenius solution. In this case, it is possible that the solution will have logarithmic terms. (Previous lecture.)

The discussion in the Course Notes – Section 1.5, pp. 39-55 – is quite readable and thorough, so there is no point in duplicating much of the information presented there. We shall, however, identify some main points.

First of all, the Frobenius method will always be able to generate one solution. For $p \geq 0$, let us return to Eq. (119). First of all, we use the fact that

$$(n+p)^2 - p^2 = (n+p+p)(n+p-p) = (n+2p)n \tag{124}$$

to rewrite (119) as

$$n(n+2p)a_n + a_{n-2} = 0\,. \tag{125}$$

Setting $n = 1$ gives

$$(1+2p)a_1 + a_{-1} = 0\,. \tag{126}$$

Since $a_1 = 0$, it follows that $a_1 = 0$. We now rewrite (125) as

$$a_n = -\frac{1}{n(n+2p)}a_{n-2}, \quad n \geq 2, \tag{127}$$

This recursion relation implies that

$$a_1 = a_3 = a_5 = \cdots = 0\,. \tag{128}$$

Let us now examine the first few even-indexed coefficients.

1. $n = 2$: $a_2 = -\dfrac{1}{2(2+2p)}a_0 = -\dfrac{1}{2^2(1+p)}$.
2. $n = 4$: $a_4 = -\dfrac{1}{4(4+2p)}a_2 = \dfrac{1}{2^5(1+p)(2+p)}$.
3. $n = 6$: $a_6 = -\dfrac{1}{6(6+2p)}a_2 = -\dfrac{1}{3 \cdot 2^7(1+p)(2+p)(3+p)}$.

There is a pattern here. In general, the Frobenius series is given by (see page 40 of the AMATH 351 Course Notes by J. Wainwright)

$$y_1(x) = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} k!(1+p)\cdots(k+p)} x^{2k} \right], \tag{129}$$

with a_0 arbitrary. The series converges for all $x \in \mathbf{R}$.

Bessel functions $J_n(x)$ of the first kind of integer order

In the case that $p = n = 0, 1, 2, \cdots$, the convention is to set the constant a_0 to be

$$a_0 = \frac{1}{2^n n!}, \tag{130}$$

to produce the (official) Bessel functions of the first kind of order n, denoted as $J_n(x)$. After some algebra, the Frobenius series obtained earlier becomes

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k}. \tag{131}$$

We write out the first few terms of the cases $n = 0$ and $n = 1$, which occur in many applications,

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots \tag{132}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \cdots. \tag{133}$$

The graphs of these two functions are shown in the figure below. (They were computed from the power series expansions.) One can also show from the power series expansions that

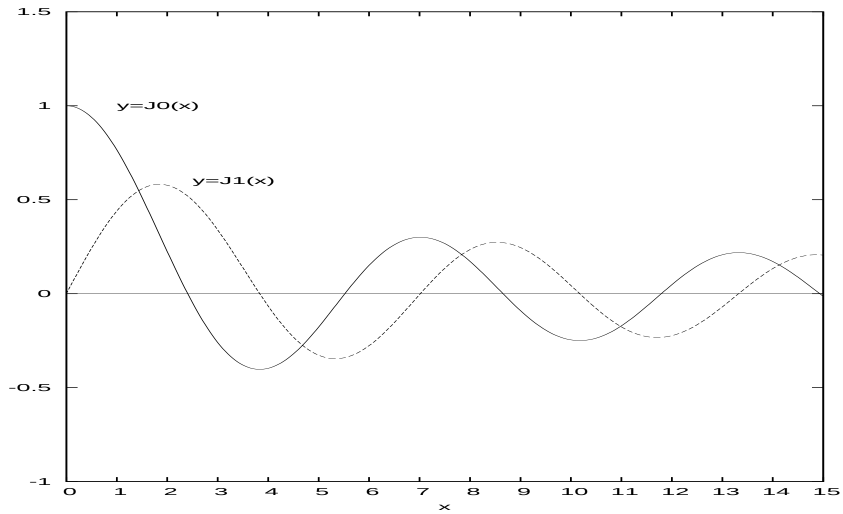
$$J_0'(x) = -J_1(x), \tag{134}$$

which implies that the zeros of J_1 are critical points, i.e. local maxima/minima, of J_0.

Bessel functions $J_p(x)$ of the first kind for arbitrary order

To define these standard functions, one assigns the value

$$a_0 = \frac{1}{2^p \Gamma(p+1)}, \tag{135}$$



where $\Gamma(x)$ denotes the Gamma function, defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{136}$$

The Gamma function is discussed in the Course Notes on pages 42-43. Its most important properties are

1. $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

2. $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

From the above, we could write that

$$\Gamma(x+1) = x! \quad \text{for} \quad x \geq 0\,, \tag{137}$$

where the function $f(x) = x!$ for $x \geq 0$ is a **continuous interpolation** of the factorial function $f(n) = n!$ defined on the nonnegative integers. One noteworthy value of the Gamma function is (see p. 42 of AMATH 231 Course Notes by J. Wainwright)

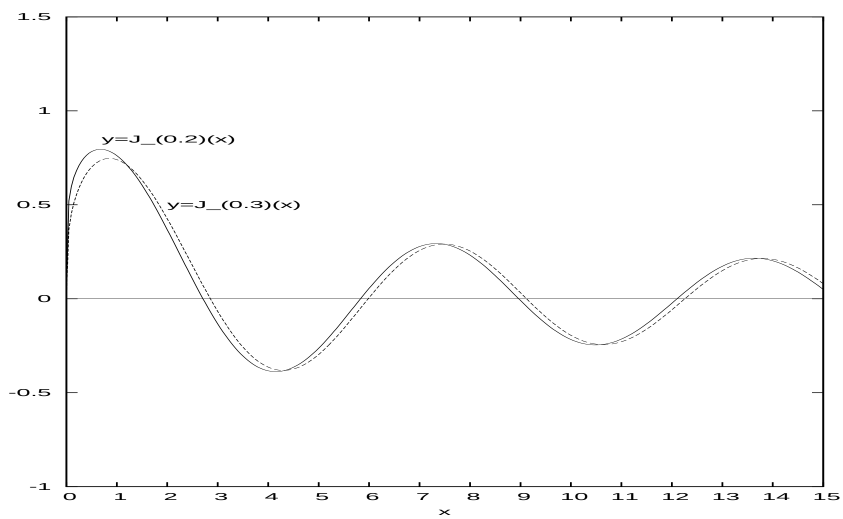
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,. \tag{138}$$

The Bessel functions $J_p(x)$ are then given by

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k}. \tag{139}$$

In the special case, $p = n$, integer, the formulas for $J_n(x)$ are obtained. This follows from the fact that $\Gamma(k+n+1) = (k+n)!$.

From a closer look at Eq. (139), one might expect that there might be some kind of continuity property of the functions $J_p(x)$ with respect to the parameter p, i.e., small changes in p will produce small changes in the function $J_p(x)$. This is shown in the figure below where the graphs of two Bessel functions with p values fairly close to each other, i.e., $p_1 = 0.2$ and $p_2 = 0.3$, are plotted. Each function may be viewed as a small perturbation of the other.



Asymptotic behaviour of $J_p(x)$ as $x \to 0^+$ and $x \to \infty$

The above series expansion shows that

$$J_p(x) \approx \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p, \quad \text{as } x \to 0^+. \tag{140}$$

Near the beginning of this lecture, cf. Eq. (116), we showed that solutions to Bessel's equation behave asymptotically as

$$y(x) \approx \frac{1}{\sqrt{x}}(c_1 \sin x + c_2 \cos x), \quad \text{as } x \to \infty. \tag{141}$$

The latter asymptotic properties are very important in quantum mechanics when the scattering of particles by other particles is studied.

Some identities satisfied by the $J_p(x)$

The Bessel functions of the first kind satisfy some interesting identities that can be very useful in practical calculations. We simply state them here and refer the reader to the Course Notes, pp. 43-44:

$$J_{p+1}(x) = \frac{p}{x} J_p(x) - J_p'(x), \tag{142}$$

$$J_{p-1}(x) = \frac{p}{x} J_p(x) + J_p'(x), \tag{143}$$

$$J_{p+1}(x) + J_{p-1}(x) = \frac{2p}{x} J_p(x). \tag{144}$$

These are examples of *three-term recurrence relations*, a general feature of *special functions* that include the Bessel functions, Legendre functions, Hermite functions, Laguerre functions. These recurrence relations play an important role in applications, including "selection rules" in quantum mechanics which tell which types of transitions from one energy state to another are allowed.

Bessel functions $J_{n+\frac{1}{2}}(x)$, n an integer

In these special cases, the Bessel functions can be expressed in terms of trigonometric functions. And in spite of the fact that $r_1 = n + \frac{1}{2}$ and $r_2 = -r_1$ so that $2p = 2n + 1$ is an integer, the Frobenius method can produce two linearly independent solutions. Here we present the following results and refer the reader to page 46 of the Course Notes:

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \tag{145}$$

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0. \tag{146}$$

You have already seen these results for $p = \dfrac{1}{2}$, but in "unscaled" form, in Eq. (111).

Bessel functions $J_p(x)$, p is not an integer

In this case, we can replace p by $-p$ in (139) to obtain a second solution:

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{k=0}^{\infty} \frac{(-1)k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k}. \tag{147}$$

Note that

$$\lim_{x\to 0^+} J_p(x) = 0\,, \qquad \lim_{x\to 0^+} J_{-p}(x) = +\infty\,, \tag{148}$$

which implies that $J_p(x)$ and $J_{-p}(x)$ are linearly independent (as expected). The general solution of Bessel's equation is then

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x)\,. \tag{149}$$

Bessel functions of the second kind, $Y_p(x)$

In the case that p is an integer $p = n \geq 0$, we have been able to construct only one class of solution to Bessel's equation, the Bessel functions $J_p(x)$ of the first kind. One can use reduction of order to construct a second, linearly independent solution.

There is a standard class of such solutions, known as Bessel functions of the second kind, and denoted as $Y_n(x)$. The most noteworthy property that distinguishes them from the Bessel functions $J_p(x)$ of the first kind is that they diverge as $x \to 0^+$:

$$Y_n(x) \to -\infty \text{ as } x \to 0^+. \tag{150}$$

In particular,

$$Y_0(x) \approx \frac{2}{\pi} \ln\left(\frac{x}{2}\right) \text{ as } x \to 0^+, \tag{151}$$

$$Y_1(x) \approx -\frac{2}{\pi} x^{-1} \text{ as } x \to 0^+. \tag{152}$$

These functions are discussed in a little more detail in the Course Notes, Section 1.5.2, pp. 46-50. The graphs of $Y_0(x)$ and $Y_1(x)$ are sketched in Figure 1.8 on page 49.

Summary: General solutions of the Bessel equation

If p is not an integer, then the general solution of Bessel's DE of order p can be written in the form

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x). \tag{153}$$

If $p = n$, a positive integer, then the general solution has the form

$$y(x) = c_1 J_n(x) + c_2 Y_n(x). \tag{154}$$

In both cases, the second, linearly independent solutions, $J_{-p}(x)$ or $Y_n(x)$, are not continuous at $x = 0$. This is the price that is paid for constructing solutions about singular points. In many physical applications, it is necessary that the solution $y(x)$ be finite at $x = 0$, which will necessitate that $c_2 = 0$.

A remarkable property of Bessel's DE: Many DEs can be transformed into it

Here we simply mention a result that is discussed in more detail in the AMATH 351 Course Notes by J. Wainwright, Section 1.1.3, pp. 7-9. It is shown (Example 1.1) that any solution of the following DE in $y(x)$,

$$x^2 y'' + (1 + 2c)xy' + (a^2 x^{2b} + c^2 - b^2 p^2)y = 0\,, \tag{155}$$

where a, b and c are constants, is of the form

$$y(x) = x^{-c} w\left(\frac{a}{b} x^b\right)\,, \tag{156}$$

where $w(z)$ is a solution of Bessel's DE,

$$z^2 w'' + zw' + (z^2 - p^2)w = 0\,. \tag{157}$$

In Problem Set 1, Question 4 of the same Course Notes (p. 196), you are asked to prove that any solution of the following DE in $y(x)$,

$$x^2 y'' + (1 + 2cx)xy' + [(a^2 + c^2)x^2 + cx - p^2]y = 0\,, \tag{158}$$

is of the form

$$y(x) = e^{-cx} w(ax)\,, \tag{159}$$

where $w(z)$ is a solution of Bessel's DE in (157).