Lecture 2

Some theory of linear second order differential equations

Let us now return to the general form of a second order linear ODE, expressed in standard form,

$$y'' + P(x)y' + Q(x)y = R(x). \tag{1}$$

For convenience, we shall also write the homogeneous ODE associated with (1),

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

Of course, Eq. (2) is a special case of Eq. (1), with $R(x) = 0$.

It would be nice to establish some conditions that guarantee the existence and uniqueness of solutions to these ODEs. In many applications, $P(x)$, $Q(x)$ and $R(x)$ are continuous on an interval of interest, except perhaps at isolated points. (We'll have to deal with such situations later.)

Existence and uniqueness of solutions to initial value problems

Theorem: Suppose that P, Q and R are continuous for all $x \in [a, b]$. Let $x_0 \in [a, b]$. Then for any pair of real numbers A, B, there exists a unique solution to Eq. (1) which satisfies the initial conditions

$$y(x_0) = A, \quad y'(x_0) = B. \tag{3}$$

A proof of this theorem is postponed until later. We shall rely on the theory of first order linear systems of DEs to prove this result. In the next section, we shall show that the **general solution** y_G of Eq. (1) can be used to solutions to the IVP in (3) for all A and B.

The conditions for existence-uniqueness assumed in the above theorem were satisfied by most, if not all, the DEs that you encountered in AMATH 250 or equivalent. For there, you studied linear second order DEs with constant coefficients, i.e.,

$$y'' + py' + qy = r(x), \tag{4}$$

where $r(x)$ is continuous. Obviously the functions $P(x) = p$ and $Q(x) = q$ are continuous everywhere. Existence and uniqueness of solutions is therefore guaranteed.

Problems can occur, however, if the conditions of the above theorem are not satisfied. For example, consider Bessel's equation with order $p = 0$,

$$x^2y'' + xy' + x^2y = 0\,, \quad x \geq 0. \tag{5}$$

We must first rewrite this DE in standard form by dividing by x^2:

$$y'' + \frac{1}{x}y' + y = 0 \quad x \neq 0\,. \tag{6}$$

Here, $P(x) = 1/x$ and $Q(x) = 1$. For any interval $I = [a, b]$ that does not contain the origin, P and Q are continuous on I so that the conditions of the above theorem are satisfied. Hence existence-uniqueness of solutions is guaranteed at any $x_0 \in I$.

But $P(x)$ is not continuous at $x = 0$. Let us investigate the nature of solutions at $x = 0$. (The following is a modification of the presentation in class.) To do so, let us first factor the above DE as follows,

$$x(xy'' + y' + xy) = 0. \tag{7}$$

This implies that for $x \neq 0$,

$$xy''(x) + y'(x) + xy(x) = 0\,, \tag{8}$$

where we have explicitly indicated the dependence of y on x. We now take the limit $x \to 0^+$ of both sides of Eq. (8) with the assumption that the limits,

$$\lim_{x\to 0^+} y''(x) \quad \text{and} \quad \lim_{x\to 0} y(x)\,, \tag{9}$$

are bounded. (More on this later.) It then follows that

$$\lim_{x\to 0^+} y'(x) = 0\,. \tag{10}$$

Assuming that $y'(x)$ is continuous (or at least right continuous) at $x = 0$, we have that

$$\lim_{x\to 0^+} y'(x) = y'\left(\lim_{x\to 0^+} x\right) = y'(0) = 0\,. \tag{11}$$

(In class, we obtained this result by simplying substituting the value $x = 0$ into (8), which is not rigorously correct.) This result implies that all solutions $y(x)$ that exist at $x = 0$ must have zero derivative there. Therefore there exist no solutions that satisfy the initial condition $y'(0) = B$ for $B \neq 0$. In conclusion, the "existence" part of "existence-uniqueness" fails to hold at $x = 0$ because of the discontinuity of $P(x)$ at $x = 0$.

Let us now return to the assumption that was made regarding the behaviour of $y''(x)$ and $y(x)$ in (9). It turns out that there is another set of solutions of Bessel's DE for which these limits do not exist, i.e., they "blow up" at $x = 0$. This clearly violates the existence-uniqueness property.

Points at which $P(x)$ or $Q(x)$ fail to be continuous are called **singular points**. It is a fact of life that we must deal with such points in many practical applications. This will come later.

Note: As mentioned in an earlier lecture, initial value problems are not the only conditions that can be applied to solutions of an ODE. In applications, it is often necessary to apply boundary value conditions of the form

$$y(x_0) = A, \quad y(x_1) = B, \quad \text{for} \quad x_0, x_1 \in I, \quad x_0 \neq x_1. \tag{12}$$

The situation for existence and uniqueness of solutions to BVPs is much more complicated and will not be discussed here.

General solutions to linear second order homogeneous DEs

Our ultimate goal is to establish the existence of a general solution to the inhomogeneous ODE,

$$y'' + P(x)y' + Q(x)y = R(x). \tag{13}$$

From this general solution, we can generate all solutions to (13). To accomplish this, we must examine the notion of a general solution to the homogeneous DE associated with (13),

$$y'' + P(x)y' + Q(x)y = 0. \tag{14}$$

Example: It will help to recall the simple example stated in Lecture 1, i.e., the ODE,

$$y'' + y = \cos 2x \,. \tag{15}$$

The general solution of this ODE is

$$y_G(x) = c_1 \cos x + c_2 \sin x - \frac{1}{3} \cos 2x \,. \tag{16}$$

As you will recall, $y_G(x)$ is composed of two parts, i.e.,

$$y_G(x) = y_h(x) + y_p(x) \,, \tag{17}$$

where

$$y_h(x) = c_1 \cos x + c_2 \sin x \,, \tag{18}$$

is the general solution of the homogeneous ODE,

$$y'' + y = 0 \,, \tag{19}$$

and

$$y_p(x) = -\frac{1}{3} \cos 2x \,, \tag{20}$$

is the particular solution of the inhomogeneous ODE (15).

Note that the general solution y_h is a **two-parameter** family of functions. In fact, y_h is a linear combination of two **linearly independent** functions,

$$y_1(x) = \cos x \,, \qquad y_2(x) = \sin x \,, \tag{21}$$

each of which is a solution to the homogeneous ODE (19).

Superposition principle for linear homogeneous DEs

The above example actually illustrates an important property of solutions to linear homogeneous DEs, namely the **superposition principle**:

Theorem: If $y_1(x)$ and $y_2(x)$ are any two solutions of the linear homogeneous DE (14), then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{22}$$

is also a solution to (14) for any values $c_1, c_2 \in \mathbf{R}$.

The proof is easy! Just "plug in"!

Proof:

$$\begin{aligned} y'' + Py' + Qy &= [c_1y_1 + c_2y_2]'' + P[c_1y_1 + c_2y_2]'' + Q[c_1y_1 + c_2y_2] \\ &= c_1[y_1'' + Py_1' + Qy_1] + c_2[y_2'' + Py_2' + Qy_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0. \end{aligned}$$

Note that the superposition principle is made possible by the linearity of the DE. Also note that the above theorem holds for $c_1 = c_2 = 0$. In other words, $y = 0$ is also a solution of the homogeneous DE, as is trivially seen by substitution. The fact that

1. $y = 0$ is a solution to ((14), and
2. any linear combination of solutions $c_1y_1 + c_2y_2$ is also a solution to (14),

suggests that **the solution space of a homogeneous DE – the set of all solutions to (14) – forms a linear vector space of functions.** In AMATH 250, and the example shown earlier, you that general solutions of homogeneous DEs are composed of two arbitrary parameters. The two-dimensional nature of the solution space will be established later. For the moment, we shall simply state that the general solution of a linear homogeneous DE of the form in (14) has the form, form

$$y_h(x, c_1, c_2) = c_1y_1(x) + c_2y_2(x)\,, \tag{23}$$

provided that y_1 and y_2 are linearly independent over the interval of interest $[a, b]$. In this case, y_1 and y_2 would be a *basis* of the solution space of the DE. You can read a little more on these ideas in Section 1.2.2 in the AMATH 351 Course Notes, p. 15.

Linear independence of solutions to homogeneous DEs and the Wronskian function

We must now recall the idea of **linear independence** of two functions $f(x)$ and $g(x)$ over an interval $[a, b]$.

Definition: Two functions $f(x)$ and $g(x)$ are said to be **linearly independent** over the interval $[a, b]$ if the following equation,

$$c_1 f(x) + c_2 g(x) = 0\,, \qquad \forall\, x \in [a, b]\,, \tag{24}$$

is satisfied only if $c_1 = c_2 = 0$. If the above equation holds for some nonzero pair c_1 and c_2, then $f(x)$ and $g(x)$ are said to be **linearly dependent**, in which case $f(x)$ is a constant multiple of $g(x)$, or vice versa.

We now prove the following important result.

Theorem: Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions to the homogeneous ODE,

$$y'' + P(x)y' + Q(x)y = 0 \tag{25}$$

over the interval $[a, b]$. Then the linear combination

$$c_1 y_1(x) + c_2 y_2(x)\,, \tag{26}$$

where c_1 and c_2 are arbitrary constants, is the general solution to (25) on $[a, b]$ in the sense than any solution of (25) on $[a, b]$ can be obtained from (26) by suitable values of the arbitrary constants c_1 and c_2.

The proof will be accomplished in several stages.

First of all, let $y(x)$ be any solution to (25) for all $x \in [a, b]$. We must show that there exist particular values of c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{27}$$

for all $x \in [a, b]$. From the Existence-Uniqueness result stated earlier (for $R(x) = 0$) the solution $y(x)$ is completely determined by its value $y(x_0)$ and its derivative $y'(x_0)$ at a point $x_0 = [a, b]$. This implies that

$$\begin{aligned} y(x_0) &= c_1 y_1(x_0) + c_2 y_2(x_0) \\ y'(x_0) &= c_1 y_1'(x_0) + c_2 y_2'(x_0)\,. \end{aligned} \tag{28}$$

This is a linear system of equations in the unknowns c_1 and c_2. For a unique solution to exist, it is sufficient that the determinant D of the system be nonzero, i.e.,

$$D = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0\,. \tag{29}$$

This leads us to investigate the following function of x,

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)\,, \tag{30}$$

which is known as the **Wronskian** of the two functions $y_1(x)$ and $y_2(x)$. In particular, we wish to examine whether or not the Wronskian $W(y_1, y_2)$ vanishes at any point $x_0 \in [a, b]$. Such points could be problematic. As we show below, it will not matter which point $x_0 \in [a, b]$ we consider.

Lemma: Let $y_1(x)$ and $y_2(x)$ be any two solutions of the homogeneous DE (25) on $[a, b]$. Then their Wronskian function $W(x) = W(y_1, y_2)$ is either identically zero or never zero on $[a, b]$.

Proof: Let us compute the derivative W' of the Wronskian of y_1 and y_2:

$$\begin{aligned} W'(x) &= [y_1(x)y_2'(x) - y_1'(x)y_2(x)]' \\ &= y_1'(x)y_2'(x) + y_1(x)y_2''(x) - y_1''(x)y_2(x) - y_1'(x)y_2'(x) \\ &= y_1 y_2''(x) - y_1''(x)y_2(x)\,. \end{aligned} \tag{31}$$

By assumption, y_1 and y_2 are solutions of the DE (25), i.e.,

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \tag{32}$$

and

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0\,. \tag{33}$$

If we multiply the first equation by y_2 and the second by y_1 and subtract the first from the second, we obtain,

$$y_1y_2'' - y_1''y_2 + P(x)(y_1y_2' - y_1'y_2) = 0\,. \tag{34}$$

But from our previous results, this equation may be rewritten in terms of $W(x)$,

$$\frac{dW}{dx} + P(x)W = 0\,. \tag{35}$$

This is a linear first order DE in the function $W(x)$ with solution (Exercise),

$$W(x) = Ae^{-\int P(x)dx}\,, \tag{36}$$

where A is an arbitrary constant. Since the exponential function is never zero, $W(x)$ is either always zero, or it is never zero. The proof of the Lemma is complete.

We are almost there, but still have to establish a relationship between the Wronskian and linear dependence/independence of two functions.

Lemma: Let $y_1(x)$ and $y_2(x)$ be two solutions to the homogeneous DE (25) on an interval $[a, b]$. Then y_1 and y_2 are linearly dependent if and only their Wronskian $W(y_1, y_2)$ is identically zero on $[a, b]$.

Proof: First assume that y_1 and y_2 are linearly dependent on $[a, b]$. From the definition of linear dependence, if one of the functions, say y_1, is identically zero on $[a, b]$ then the other function must also be identically zero, and the Wronskian automatically vanishes. In the case that one function, say y_1, does not vanish identically, it follows, from the definition of linear

dependence, that $y_2 = cy_1$ for some constant c. The Wronskian $W(y_1, y_2)$ then becomes

$$\begin{aligned} W(y_1, y_2) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= y_1(x)[cy_1(x)]' - y_1'(x)[cy_1(x)] \\ &= c[y_1(x)y_1'(x) - y_1'(x)y_1(x)] \\ &= 0\,. \end{aligned} \tag{37}$$

Now assume that the Wronskian $W(y_1, y_2)$ is identically zero on $[a, b]$. We must prove linear dependence of y_1 and y_2.

From our earlier discussion, if one of the functions, say y_1, is identically zero on $[a, b]$, then two functions are linearly dependent. We now assume that y_1 is not identically zero on $[a, b]$, which means that there is a subinterval $[c, d] \subseteq [a, b]$ such that $y_1(x) \neq 0$ for all $x \in [c, d]$. Since the Wronskian W is assumed to be identically zero on $[c, d] \subseteq [a, b]$, we can divide by $[y_1(x)]^2 \neq 0$ to obtain,

$$\begin{aligned} \frac{W(y_1, y_2)}{y_1^2} &= \frac{y_1(x)y_2'(x) - y_1'(x)y_2(x)}{[y_1(x)]^2} \\ &= \left[\frac{y_2(x)}{y_1(x)}\right]' \\ &= 0\,. \end{aligned} \tag{38}$$

This implies that

$$\frac{y_2(x)}{y_1(x)} = K\,, \tag{39}$$

for some constant K, or

$$y_2(x) = Ky_1(x)\,, \quad x \in [c, d]\,. \tag{40}$$

Therefore,

$$y_2'(x) = Ky_1'(x)\,, \quad x \in [c, d]\,. \tag{41}$$

Recalling that $y_1(x)$ and $y_2(x)$ are solutions to (25), it is sufficient, from the Existence-Uniqueness Theorem stated at the beginning of this section, to conclude that $y_1(x)$ and $Ky_2(x)$ are the same solution to (25) for all $x \in [a, b]$, i.e.,

$$y_2(x) = Ky_1(x) \qquad x \in [a, b]\,. \tag{42}$$

Therefore $y_1(x)$ and $y_2(x)$ are linearly dependent on $[a, b]$ and the Lemma is proved.

From the above two Lemmas, we have proved the Theorem stated earlier.

Let's return to our earlier example, namely, the homogeneous DE,

$$y'' + y = 0\,. \tag{43}$$

We claimed that the general solution to this DE was

$$y_h(x) = c_1 \cos x + c_2 \sin x \tag{44}$$

at all $x \in \mathbb{R}$. Letting

$$y_1(x) = \cos x\,, \quad y_2(x) = \sin x\,, \tag{45}$$

we compute their Wronskian to be

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= \cos^2 x + \sin^2 x \\ &= 1 \\ &\neq 0\,. \end{aligned} \tag{46}$$

So, indeed, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two linearly independent solutions to the DE over an interval $[a, b]$ on real line $\mathbb{R}$ (including the entire real line) and $y_h(x)$ is the general solution.

Lecture 3

General solutions of linear inhomogeneous DEs

We are now in a position to discuss the general solution of a inhomogeneous DE of the form,

$$y'' + P(x)y' + Q(x)y = R(x)\,, \tag{47}$$

over an interval $[a, b]$. As shown below, we'll use the general solution y_h of the associated homogeneous DE,

$$y'' + P(x)y' + Q(x)y = 0\,, \tag{48}$$

to construct the general solution to (47). It will be a two-parameter family of functions that we shall denote as $y_G(x, c_1, c_2)$. From y_G we can then construct all solutions of (47) over an interval $[a, b]$. This means that for a point $x_0 \in [a, b]$ and any pair of values $A, B \in \mathbb{R}$, there exist unique values of c_1 and c_2 – let's call them C_1 and C_2, respectively – such that

$$y_G(x_0, C_1, C_2) = A, \qquad y_G'(x_0, C_1, C_2) = B. \tag{49}$$

Theorem: Let $y_h(x, c_1, c_2)$ denote the general solution to the homogeneous DE in (48). Recall that it will have the form,

$$y_h(x, c_1, c_2)) = c_1 y_1(x) + c_2 y_2(x), \tag{50}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (48) over $[a, b]$. Moreover, let $y_p(x)$ be a solution to the inhomogeneous DE in (47). Then the general solution to (47) is given by

$$y_G(x, c_1, c_2) = y_p(x) + y_h(x, c_1, c_2). \tag{51}$$

Before we prove this result, you might ask, "How did we get $y_p(x)$?" The answer is, "We don't care how!" We're just assuming that somehow, we were given not only y_p but y_h, the general solution to the homogeneous DE.

Proof: Let $y(x)$ be *any* solution to the inhomogeneous DE in (47). Now define the function

$$u(x) = y(x) - y_p(x). \tag{52}$$

Let's now substitute $u(x)$ into the LHS of (48),

$$\begin{aligned} u'' + Pu' + Qu &= [y - y_p]'' + P[y - y_p]' + Q[y - y_p] \\ &= [y'' + Py' + Qy] - [y_p'' + Py_p' + Ry_p] \\ &= R(x) - R(x) \\ &= 0, \end{aligned}$$

where we have used the assumptions that y and y_p are solutions to DE (48). It follows that u is a solution to the homogeneous DE (48). Since it is a solution to (48), one must be able to construct it from the general solution y_h of (48), which we assume that we know. In other words, there will exist particular values of c_1 and c_2 for which

$$y_h(x, c_1, c_2) = u(x) = y(x) - y_p(x). \tag{53}$$

Rearranging, we have

$$y(x) = y_p(x) + y_h(x, c_1, c_2) \tag{54}$$

and the theorem is proved.

There is an importance consequence of this theorem:

The generality of the general solution y_G of the inhomogeneous DE in (47) is contained in y_h, the general solution of the associated homogeneous DE (48).

As a consequence, in order to determine the general solution of an inhomogeneous DE of the form (47), we need to find the following,

1. a particular solution $y_p(x)$ to Eq. (1), and
2. the general solution $y_h(x)$ to Eq. (2).

Of course, you were using this principle in your previous course on DEs, e.g., AMATH 250/251, when you were constructing general solutions to second order DEs with constant coefficients. But this principle now applies to all linear second order ODEs.

Homogeneous DEs - Using a known solution to find another using "Reduction of Order"

From the previous section, we know that general solutions to the homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{55}$$

may be constructed from any two linearly independent solutions y_1 and y_2 of (55). In general, however, it is not trivial to find such linearly independent solutions. However, if you do happen to know one solution, y_1, you can find another solution by a method known as the **Reduction of Order**.

Recall that if y_1 is a solution to (55) then cy_1, for any constant $c \in \mathbf{R}$, is also a solution. The idea is to replace the constant c by a function u to be determined by demanding that y_2, defined as $y_2(x) = u(x)y_1(x)$, is a solution to (55). First substitute y_2 into (55):

$$(uy_1)'' + P(uy_1)' + Quy_1 = 0. \tag{56}$$

Performing the differentiation,

$$u''y_1 + 2u'y_1' + uy_1'' + Pu'y_1 + Puy_1' + Quy_1 = 0. \tag{57}$$

But the LHS can be rearranged to

$$u''y_1 + 2u'y_1' + Pu'y_1 + u(y_1'' + Py_1' + Qy_1) = u''y_1 + (2y_1' + Py_1)u' = 0, \tag{58}$$

where we have used the fact that y_1 is a solution to (55). Now let $v = u'$ so that the above DE becomes a first-order DE in v (hence the term *reduction of order*):

$$v'y_1 + (2y_1' + Py_1)v = 0. \tag{59}$$

We may rearrange this DE as follows,

$$\frac{v'}{v} = -2\frac{y_1'}{y_1} - P \tag{60}$$

which can be integrated to give

$$\ln v = -2\ln y_1 - \int P(x)dx. \tag{61}$$

Exponentiating gives

$$u' = v = \frac{A}{y_1^2} e^{-\int P(x)dx}. \tag{62}$$

Without loss of generality, we set $A = 1$ so that the final result is

$$u(x) = \int \frac{1}{y_1^2(x)} e^{-\int P(x)dx} dx \tag{63}$$

Using Wronskians, one can show (Exercise) that y_1 and $y_2 = uy_1$ are linearly independent.

Note: It is not recommended to memorize the formula in Eq. (63). Simply recall the basic idea: If you know one solution, $y_1(x)$, then assume the second solution to have the form $y_2(x) = u(x)y_1(x)$. Substitute this form into the DE of concern and let the math take you to the first order differential equation in $v = u'$ that you must solve. And then go from there.

Example 1: This is a "test run" on a simple case, the DE,

$$y'' + y = 0\,. \tag{64}$$

Of course, we know that two linearly independent solutions to this DE are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Let's see what happens if we pretend that we do not know $y_2(x)$, and use $y_1(x)$ to find it. In other words, we assume a second solution of the form

$$y_2(x) = u(x)y_1(x) = u(x)\cos x\,. \tag{65}$$

Computing the necessary derivatives,

$$y_2' = u'\cos x - u\sin x\,, \tag{66}$$

$$y_2'' = u''\cos x - 2u'\sin x - u\cos x\,. \tag{67}$$

Substitution into the DE yields,

$$y_2'' + y_2 = u''\cos x - 2u'\sin x - u\cos x + u\cos x = 0\,, \tag{68}$$

or

$$u''\cos x - 2u'\sin x = 0\,. \tag{69}$$

Now let $v = u'$ so that the above DE becomes,

$$v' \cos x - 2v \sin x = 0\,. \tag{70}$$

This is a separable first order DE in v:

$$\frac{v'}{v} = 2\,\frac{\sin x}{\cos x}\,. \tag{71}$$

Integration yields,

$$\ln|v(x)| = -2\ln|cosx| + C\,. \tag{72}$$

Exponentiation yields

$$u'(x) = v(x) = \frac{A}{\cos^2 x} = A\sec^2(x)\,. \tag{73}$$

Integration yields,

$$u(x) = A\tan x + B\,. \tag{74}$$

Without loss of generality, we may set $A = 1$ and $B = 0$ so that $u(x) = \tan x$, so that

$$y_2(x) = u(x)\cos x = \tan x \cos x = \sin x\,, \tag{75}$$

as expected.

Example 2: By "inspection" $y_1 = x^2$ is a solution to the homogeneous DE

$$x^2y'' + xy' - 4y = 0. \tag{76}$$

We now assume a second solution of the form $y_2 = uy_1 = ux^2$. Computing the necessary derivatives:

$$y_2' = 2xu + x^2u', \tag{77}$$

$$y_2'' = 2u + 4xu' + x^2u''. \tag{78}$$

Substitution into the DE yields

$$2x^2u + 4x^3u' + x^4u'' + 2x^2u + x^3u' - 4ux^2 = 0, \tag{79}$$

which simplifies to

$$x^4u'' + 5x^3u' = 0. \tag{80}$$

(Note how the fact that y_1 is a solution to the DE automatically produces the cancellation of terms that simplifies the above DE.) Assuming that $x \neq 0$, we may rewrite this separable DE as follows,

$$\frac{u''}{u'} = -\frac{5}{x}. \tag{81}$$

We can set $v = u'$ or simply integrate this DE to give

$$\ln u' = -5 \ln x, \tag{82}$$

which, upon exponentiation, gives $u'(x) = Ax^{-5}$, where A is an arbitrary constant. Integration gives $u(x) = Bx^{-4}$, where B is an arbitrary constant. However, without loss of generality, we may set $B = 1$ so that $u(x) = x^{-4}$. This yields the second solution

$$y_2 = uy_1 = \frac{1}{x^4}x^2 = \frac{1}{x^2}, \tag{83}$$

which may be verified by substitution.

You really don't need to perform a Wronskian analysis to see that the solutions $y_1 = x^2$ and $y_2 = x^{-2}$ are linearly independent on any interval $[a, b]$ not containing the point $x = 0$. Neither solution is a constant multiple of the other. Therefore, the general solution to the DE is

$$y(x) = C_1x^2 + \frac{C_2}{x^2} \tag{84}$$

on any interval $[a, b]$ not containing 0.

That being said, let's compute the Wronskian of the two solutions in order to verify the

above statement:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} x^2 & \frac{1}{x^2} \\ 2x & -\frac{2}{x^3} \end{vmatrix}, \qquad x \neq 0 \\ &= -\frac{2}{x} - \frac{2}{x}, \qquad x \neq 0 \\ &= \frac{4}{x}, \qquad x \neq 0 \\ &\neq 0, \qquad x \neq 0. \end{aligned} \tag{85}$$

The Wronskian is nonzero for all $x \neq 0$ and undefined at $x = 0$. This implies that the two functions are linearly independent on any interval $[a, b]$ that does not contain the point $x = 0$.

Lecture 4

Some linear second order differential equations in applied mathematics and theoretical physics

As you have already seen in AMATH 250, DEs are indispensible in the modelling of natural phenomena since many of the laws of nature involve rates of change. Perhaps the most familiar natural law involving a rate of change is Newton's second law, which is often referred to as

$$\mathbf{F} = m\mathbf{a}\,. \tag{86}$$

For a particle of mass m, for example, this translates to a second order DE in the position vector $\mathbf{x}(t)$:

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \mathbf{F}. \tag{87}$$

In the simplest case, $\mathbf{F}$ will be a function solely of the position $\mathbf{x}$ of the particle. Even so, this DE is solvable in closed form only for a few special forces, e.g., the harmonic oscillator and the Kepler problem (two-body gravitational force). Only the former leads to a linear DE. A particular trajectory of the particle is usually extracted by imposing the following initial conditions on its position and velocity:

$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}'(0) = \mathbf{v}(0) = \mathbf{v}_0. \tag{88}$$

This is an **initial value problem** that determines the future dynamics of the particle.

There are many other phenomena that are described by differential equations. Not surprisingly, much of theoretical physics is formulated in terms of differential equations, most often partial differential equations (PDEs). As we'll see below, many of these PDEs give rise to linear second order ODEs, the subject of this course. A few examples of these PDEs are listed below. In what follows, ∇^2 denotes the Laplacian operator which, in Cartesian coordinates, is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{89}$$

In the following, u and ψ are scalar-valued functions in $\mathbb{R}^3$.

1. **Laplace's equation:**

$$\nabla^2 u = 0.$$

   Electrostatics, hydrodynamics, heat flow, gravitation.

2. **Poisson's equation:**

$$\nabla^2 = -\rho/\epsilon_0.$$

   Electrostatics.

3. **Wave equation:**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \nabla^2 u.$$

   Propagation of waves.

4. **Heat/diffusion equation:**

$$\frac{\partial u}{\partial t} = D\nabla^2 u.$$

5. **Time-dependent Schrödinger wave equation:**

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial \psi}{\partial t}$$

   and the **time-independent Schrödinger wave equation:**

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi.$$

   Quantum mechanics.

Obviously the main subject of this course is ordinary differential equations, *not* partial differential equations. (That is the subject of AMATH 353.) It is nevertheless important to see how these standard PDEs give rise to some of the well-known second order linear ODEs, e.g., the Bessel, Legendre, Hermite, Laguerre DEs that we shall see in this course. In what follows, we shall examine a couple of examples.

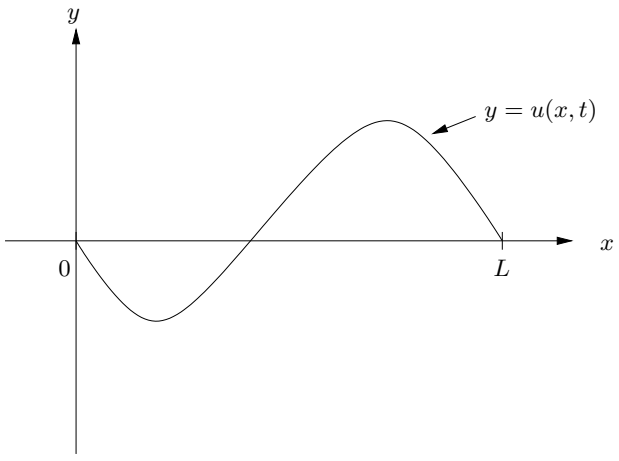
The vibrating string and the one-dimensional wave equation

Here we consider a taut horizontal, thin string of length L that is clamped at both ends. The tension T in the string is assumed to be constant. We shall also assume that the string is homogeneous. The problem is to set up an equation that will describe vibrations of the string – vertical motions that would result, for example, by initially plucking the string (as in the case of a guitar string).

We assume that each point on the string may be identified by its x-coordinate, where $x \in [0, L]$. At rest, the string lies on an x-axis of a Cartesian coordinate system in the plane. During the vibration of the string, each point $x \in [0, L]$ of the string is assumed to undergo a purely vertical displacement from the x-axis. We shall let $u(x, t)$ denote the vertical displacement of this point at time $t \geq 0$. The variation in u with respect to time t at a point $x \in [0, L]$ will represent the vibration of the string at point x. The variation in u with respect to x at time t will represent the shape of the vibrating string at time t. Notice that we shall have to impose boundary conditions on $u(x, t)$ because of the clamping at both ends:

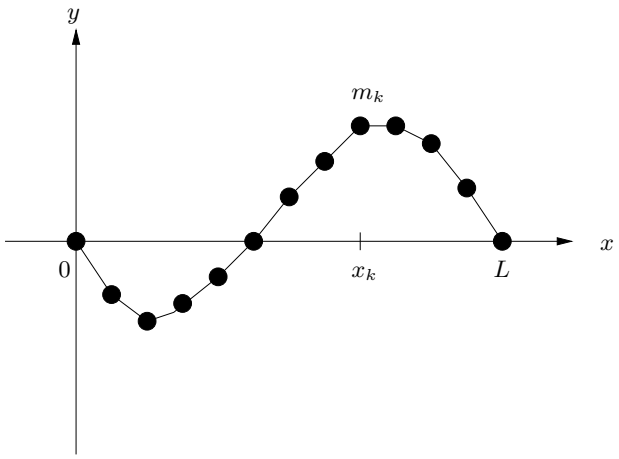
$$u(0, t) = u(L, t) = 0 \quad \text{for all } t \geq 0. \tag{90}$$

A "snapshot" of the vibrating string at a time $t > 0$ will produce a profile such as the one sketched below.

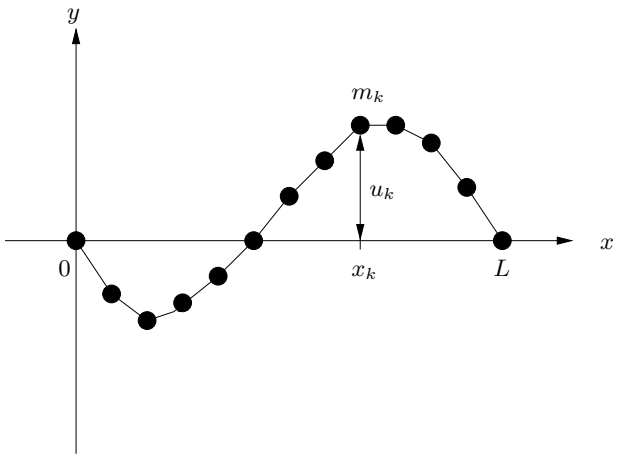


In the "Spirit of Calculus" we divide the string up into tiny pieces as follows. For a given $n > 0$, we define $\Delta x = L/n$ and construct the partition points $x_k = k\Delta x$ for $k = 0, 1, \cdots, n$

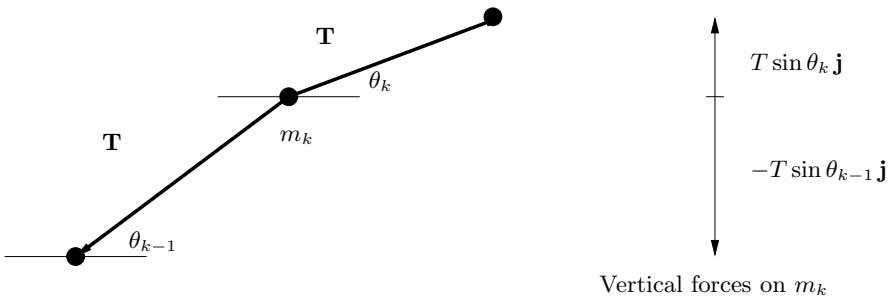
so that $x_0 = 0$ and $x_n = L$. We then approximate the string as a set of beads of equal mass $m_k = m$ that are located at the x_k and connected to each other with massless strings that have tension T:



The mass of each bead is $m = \rho\Delta x$, where ρ is the lineal density (mass per unit length) of the string, assumed to be constant. It now remains to determine the net force on each mass in order to come up with an equation of motion for the masses. For simplicity of notation, we let $u_k = u(x_k, t)$ denote the vertical displacement of each mass at a given time t, as shown below:



The net vertical force on each mass is the vector sum of the projections of the two tension vectors in the **j** direction. It is convenient to introduce the angles θ_k that each tension vector (which is directed along the line joining two consecutive masses) makes with the horizontal:



Vertical forces on m_k

The net vertical force on mass m_k shown above is

$$T\sin\theta_k - T\sin\theta_{k-1} \tag{91}$$

For small oscillations, the θ_k will be small so that we can use the approximation $\sin\theta \approx \tan\theta$. This is possible because $\cos\theta \approx 1$ for small θ. We have used the tan function because it can now be expressed in terms of the u_k's:

$$\begin{aligned} T[\sin\theta_k - \sin\theta_{k-1}] &\approx T[\tan\theta_k - \tan\theta_{k-1}] \\ &= T\left[\frac{u_{k+1}-u_k}{\Delta x} - \frac{u_k - u_{k-1}}{\Delta x}\right]. \end{aligned} \tag{92}$$

This is the net force on each bead m_k. We now use Newton's "Second Law", i.e. force = (mass)(acceleration):

$$\rho\Delta x\frac{\partial^2 u(x_k,t)}{\partial t^2} = T\left[\frac{u_{k+1}-u_k}{\Delta x} - \frac{u_k-u_{k-1}}{\Delta x}\right] \tag{93}$$

or

$$\rho\frac{\partial^2 u(x_k,t)}{\partial t^2} = T\frac{1}{\Delta x}\left[\frac{u_{k+1}-u_k}{\Delta x} - \frac{u_k-u_{k-1}}{\Delta x}\right] \tag{94}$$

Apart from the T, the expression on the right side is an approximate rate of change of an approximate rate of change of u with respect to x. Therefore, in the limit $n \to \infty$, which implies $\Delta x \to 0$, the expression on the right side becomes a second derivative. (We omit all technical details here.) Thus, in the limit $\Delta x \to 0$, we have the result

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{95}$$

or

$$\boxed{\frac{\partial^2 u}{\partial t^2} = K\frac{\partial^2 u}{\partial x^2}, \quad K = \frac{T}{\rho}.} \tag{96}$$

This is the **one-dimensional wave equation**, a partial differential equation in $u(x,t)$. Of course, PDEs are the subject of AMATH 353, but it won't take too much to find a solution to this PDE. One of the first things to try is the method of **separation of variables**: One assumes a solution of the form,

$$u(x,t) = v(x)w(t). \tag{97}$$

In fact, since we are looking for oscillatory solutions, let's go further and assume that the time-dependent function $w(t)$ is given by

$$w(t) = \cos\omega t, \tag{98}$$

where ω is to be determined. If we now substitute (97) into (96), we obtain

$$-\omega^2 v(x)\cos\omega t = Kv''(x)\cos\omega t\,. \tag{99}$$

We can divide out the $\cos\omega t$ and rearrange to obtain the following linear, second-order DE in $v(x)$,

$$\boxed{v''(x) + \frac{\omega^2}{K}v(x) = 0\,.} \tag{100}$$

Important note: It is worthwhile to mention that Eq. (100) represents an important step in our procedure to determine solutions to the wave equation PDE in (96). The method of separation of variables has reduced the problem of finding solutions to a PDE to one of finding solutions to an ODE.

We recognize Eq. (100) as the **linear harmonic oscillator DE** with general solution,

$$v(x) = c_1\cos\nu x + c_2\sin\nu x, \quad \nu = \frac{\omega}{\sqrt{K}}. \tag{101}$$

We still don't know what ω is. But let us now impose the boundary conditions of the clamped string on the above general solution. The first condition, $v(0) = 0$, implies that $c_1 = 0$. The

second condition, $v(L) = 0$ implies that (we may ignore the c_2 factor)

$$\sin(\nu L) = \sin\left(\omega\sqrt{\frac{\rho}{T}}L\right) = 0. \tag{102}$$

This implies that the argument of the sin function must be a multiple of π, i.e.,

$$\omega\sqrt{\frac{\rho}{T}}L = n\pi, \quad n = 1, 2, \cdots, \tag{103}$$

which, in turn, implies that our previously unknown ω can assume only a discrete set of values,

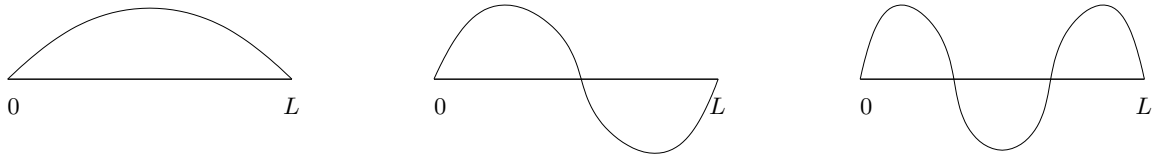
$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \tag{104}$$

These are the "natural" or fundamental frequencies of vibration of the clamped string. The spatial solutions $v_n(x)$ that correspond to these frequencies are given by

$$v_n(x) = C\sin\omega_n\sqrt{\frac{\rho}{T}}x = C\sin\left(\frac{n\pi x}{L}\right). \tag{105}$$

These are the "fundamental modes" or "harmonics" of the vibrating string. The first three modes are sketched below:

$v_1(x) = \sin\left(\frac{\pi x}{L}\right)$ $v_2(x) = \sin\left(\frac{2\pi x}{L}\right)$ $v_3(x) = \sin\left(\frac{3\pi x}{L}\right)$

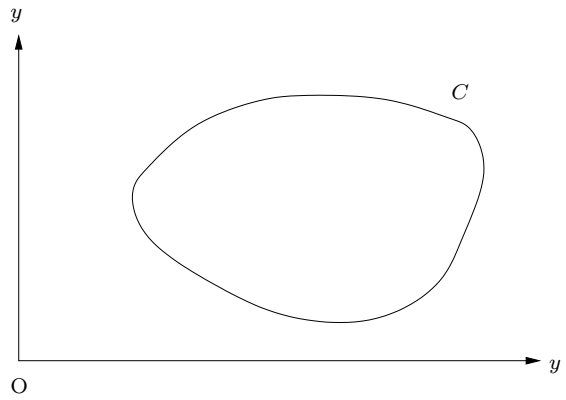


A more detailed discussion of these modes and their role in solutions of the one-dimensional wave equation associated with the vibrating string will be given in AMATH 353. In closing, let us call attention to the fact that the separation of variables method applied to the one-dimensional wave equation produced the linear second order DE in Eq. (100), the well-known harmonic oscillator equation. As we'll see below, in higher dimensions, the separation of variables method will give rise to more sophisticated linear second order DEs – those with nonconstant coefficients, the subject of this course.

Final remarks: In the derivation of the wave equation, we assumed that the string was homogeneous, with constant lineal density ρ and tension T. We can relax those assumptions to arrive at a wave equation where the constant K is replaced by an x-dependent coefficient $K(x) = T(x)/\rho(x)$ in Eq. (96). The consequence of this relaxation is that the second order DE in $v(x)$ obtained by separation of variables, cf. Eq. (100) would no longer have constant coefficients and therefore not so easily solvable. In fact, in the hanging-chain problem – see AMATH 351 Course Notes by J. Wainwrght, p. 4 – the density is still assumed to be constant but the tension function is linear in x. After separation of variables, we arrive at an equation that can be transformed into Bessel's equation, cf. p. 5.

The two-dimensional wave equation and the vibrating membrane problem

We now consider the problem of a thin, homogeneous vibrating membrane, for example, that of a drum with a circular boundary. For the moment, we are not concerned with the shape of the membrane but will just state that it occupies a bounded region D in $\mathbb{R}^2$ with boundary curve C, as sketched below.



We assume that at rest, the membrane lies on the xy plane. During vibration, it is clamped to the xy plane along the curve C. We'll let $u(x, y)$ denote the vertical displacement of the membrane during vibration. Because of the clamping condition, u will have to satisfy the boundary condition $u(x, y) = 0$ for $(x, y) \in C$.

Once again appealing to the "Spirit of Calculus," we can divide up the membrane into tiny squares and replace each square by an appropriate bead of mass Δm situated on a set of grid

points (x_k, y_k) and connected to each other by a system of massless strings with equal tension. After analyzing the forces that act on these masses for small displacements, one arrives at the following two-dimensional wave equation that describes the evolution of $u(x, y, t)$:

$$\frac{\partial^2 u}{\partial t^2} = K\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = K\nabla^2 u. \tag{106}$$

We are not restricted to the use of Cartesian coordinates for this problem, even though we used them in the derivation of the wave equation. For example, if the drum has a circular boundary (centered at the origin), then it is much more convenient to use polar coordinates. We have arrived at an important point in the solution of such physical problems: It is helpful to consider the symmetry of the problem when selecting a coordinate system to solve it.

In order to illustrate how Bessel's equation can arise from this problem, we consider a circular drum centered at the origin, with radius R_0. Using polar coordinates, we let the vertical displacement of the membrane be given by $u(r, \theta, t)$. As in the one-dimensional case, we perform a separation of variables – keeping in mind that we are interested in time-periodic solutions, we assume a solution of the form

$$u(r, \theta, t) = v(r, \theta)\cos\omega t. \tag{107}$$

Substitution into Eq. (106) yields the following second order partial differential equation for v:

$$\nabla^2 + a^2 v = 0, \qquad a^2 = \frac{K}{\omega}. \tag{108}$$

(Note that we have not yet expressed the Laplacian operator ∇ in polar coordinates. We would have obtained the above equation regardless of the coordinate system employed in the plane.) In polar coordinates, the Laplacian operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{109}$$

Substitution into (108) yields the following PDE in v:

$$\nabla^2 = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + a^2 v = 0. \tag{110}$$

This is still a PDE! We perform another separation of variables by assuming $v(r,\theta) = R(r)S(\theta)$ to give

$$R''S + \frac{1}{r}R'S + \frac{1}{r^2}RS'' + a^2RS = 0. \tag{111}$$

Now divide by RS (we won't worry about zeros here) to give

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{S''}{S} + a^2 = 0. \tag{112}$$

We now multiply through by r^2 and move the term S''/S to the other side:

$$r^2\frac{R''}{R} + r\frac{R'}{R} + a^2r^2 = -\frac{S''}{S}. \tag{113}$$

Notice that the left side of this equation depends only upon the independent variable r and the right side depends only upon the independent variable θ. The only way that this situation can be reconciled is if both sides equal a constant, call it A. Let's see the consequence of this result. From (113), we have that

$$-\frac{S''}{S} = A \quad \text{or} \quad S'' + AS = 0. \tag{114}$$

This is a second order linear DE in the function $S(\theta)$. The nature of the solutions to this DE depends upon whether A is positive or negative. We'll now argue that A must be positive, and even more.

Let's go back to the assumption under separation of variables, that $v(r,\theta) = R(r)S(\theta)$. Recall that this function represents the vertical displacement of the membrane. It must be single-valued as far as θ is concerned, i.e., $v(r,\theta + 2\pi) = v(r,\theta)$. This means that $S(\theta)$ must be periodic in θ, i.e., $S(\theta + 2\pi) = S(\theta)$. Periodic solutions of (114) are possible only if $A > 0$. Moreover, these solutions are 2π-periodic only if $A = m^2$, $m = 1, 2, \cdots$, in which case the general solution to Eq. (114) has the form

$$S(\theta) = c_1 \cos m\theta + c_2 \sin m\theta. \tag{115}$$

If we plug this result, $A = m^2$, $m = 1, 2, \cdots$, back into (113), and multiply by R we arrive at the following ODE for $R(r)$:

$$r^2R'' + rR' + (a^2r^2 - m^2)R = 0. \tag{116}$$

This is a linear second order ODE in $R(r)$ with nonconstant coefficients. We're almost there. Now make the change of variables $s = ar$. We'll have to replace derivatives w.r.t. r with derivatives w.r.t. s:

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = a\frac{dR}{ds} \tag{117}$$

and

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right)\frac{d}{ds}\left(a\frac{dR}{ds}\right)\frac{ds}{dr} = a^2\frac{d^2R}{ds^2}. \tag{118}$$

Substituting these results into (116) and simplifying yields the following DE in the function $R(s) = R(ar)$:

$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + (s^2 - m^2)R = 0. \tag{119}$$

We've arrived! This is "Bessel's equation of order m". Its general solution may be expressed in terms of two linearly independent "Bessel functions" that we shall discuss a little later in this course:

$$R(s) = c_1 J_m(s) + c_2 Y_m(s). \tag{120}$$

And by imposing the "clamping condition" on the circular drum, we will obtain the fundamental frequencies of vibration of the drum.

Note: A nice animation of the vibrating drum can be found on the first page of the Wikipedia site entitled, "Bessel functions".

In closing, we remark that the separation of variables method applied to any of the PDEs listed earlier that involve the Laplacian operator will yield second order ODEs – when Cartesian coordinates are *not* used, these ODEs will have nonconstant coefficients. The exact form of the ODE will depend upon the coordinate system used.